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Calculation of the Invariant Measures at Weak Disorder for the Two-Line Anderson Model

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Abstract

We compute the invariant measures in the weak disorder limit, for the Anderson model on two coupled chains. These measures live on a three-dimensional projective space, and we use a total set of functions on this space to characterise the measures. It turns out that at zero energy, there is a similar anomaly as first found by Kappus and Wegner for the single chain, but that, in addition, the measures take a different form on different regions of the spectrum.

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1 Introduction and formulation of the problem

In this paper we consider the invariant measure for the one-dimensional Anderson model on two coupled chains. The Hamiltonian is given by $H = H_0 + \lambda V$, where

$$(H_0\psi)(n,s) = \psi(n+1,s) + \psi(n-1,s) + \psi(n,s\pm 1) \tag{1.1}$$

and

$$(V\psi)(n,s) = v_{n,s}\psi(n,s) \tag{1.2}$$

where $s = 1,2$ and the $v_{n,s}$ are i.i.d. random variables. In the case of a single chain, this model has been studied extensively. In particular, it was proved by Goldsheid, Molchanov and Pastur [1] that the spectrum is entirely pure-point and all corresponding eigenfunctions are exponentially localised. This result was extended to the case of a strip (in particular the case of two chains) by Lacroix [2, 3] using a method proposed by Pastur [4] and a generalisation of Fürstenberg's theorem[5] due to Osseledec[6]. (For a comprehensive overview of the theory, see the book by Carmona and Lacroix [7].)

To get insight into the behaviour for small disorder, Thouless [8] attempted to write down a perturbation expansion in the disorder (i.e. in λ) of the invariant measure in the case of a single chain. In terms of the variable $Z(n) = \psi(n)/\psi(n-1)$ the Schrödinger equation at energy E for this case can be written as

$$Z(n+1) = E - \lambda v_n - \frac{1}{Z(n)}.$$

The invariant measure ν_λ^E for this transformation is then defined by

$$\int f(x)\nu_\lambda^E(dx) = \mathbb{E}\int f(E - \lambda v - \frac{1}{x})\nu_\lambda^E(dx) \tag{1.3}$$

for all bounded continuous functions f. The Liapunov exponent $\gamma(E)$ and the density of states $N(E)$ are related to this measure by

$$\gamma(E) = \mathrm{Re}\,\tilde\gamma(E); \quad N(E) = \pi\mathrm{Im}\,\tilde\gamma(E), \tag{1.4}$$

where

$$\tilde\gamma(E) = \int \ln x\, \nu_{\eta,E}(dx). \tag{1.5}$$

Kappus and Wegner [9] subsequently discovered that the perturbation series proposed by Thouless is incorrect for the case $E = 0$. They called this an *anomaly*. In fact, the limiting measure ν_0^E is discontinuous at $E = 0$. The problem was further analysed by Derrida and Gardner [10]. They found that the perturbation series is also anomalous at the values $E = 2\cos\frac{p}{q}\pi$ for integer p and q. Bovier and Klein [11] then completed their investigation and derived the correct perturbation series in all cases. These series were subsequently shown to be asymptotic by Campanino and Klein [12] by means of a very sophisticated analysis.

Here we consider the analogous problem for the case of two lines. We concentrate on the more limited objective of proving the convergence of the measures as $\eta \to 0$ and determining the limiting invariant measures. In the case of a single chain, this amounts to

$$\lim_{\lambda\downarrow 0}\nu_\lambda^E = \begin{cases} \dfrac{c}{x^2 - Ex + 1}dx & \text{if } E \neq 0; \\ \dfrac{c_0}{\sqrt{x^4+1}}dx & \text{if } E = 0. \end{cases} \tag{1.6}$$

We prove in a much simpler fashion than [12] that these limits hold in the sense of weak convergence of measures. (This result is of course much weaker than theirs.) We next generalise our approach to the case of two coupled chains. It turns out that this case is considerably more complicated. In particular, the limiting measure has a different appearance on different regions of the unperturbed spectrum. An outline of our method has been published in [13].

The unperturbed ($\lambda = 0$) spectrum for the Hamiltonian 1.1 has two branches:

$$E(k) = 2\cos k \pm 1; \qquad k \in [-\pi, \pi].$$

These dispersion relations (2.3) are depicted in Figure 1. We can write the Schrödinger equation

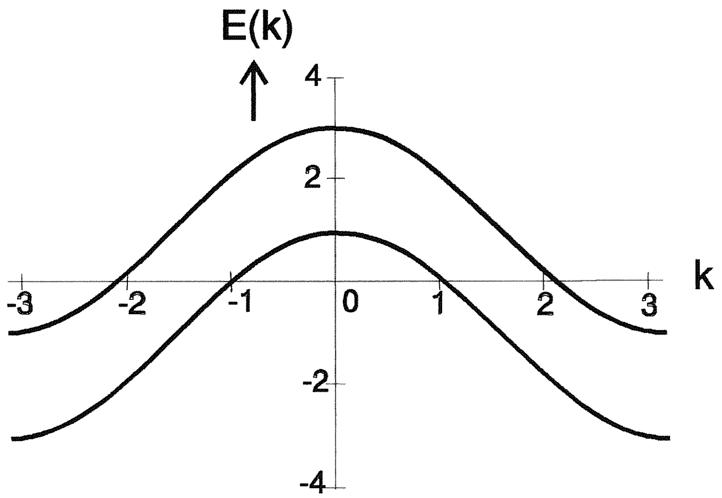

Figure 1: *The dispersion relation for two linked chains*

for this case in transfer matrix form as follows:

$$\begin{pmatrix} \psi(n+1,1) \\ \psi(n+1,2) \\ \psi(n,1) \\ \psi(n,2) \end{pmatrix} = \begin{pmatrix} E - \lambda v_{n,1} & -1 & -1 & 0 \\ -1 & E - \lambda v_{n,2} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi(n,1) \\ \psi(n,2) \\ \psi(n-1,1) \\ \psi(n-1,2) \end{pmatrix}. \tag{1.7}$$

This can be written more concisely as

$$\begin{pmatrix} \vec{\psi}(n+1) \\ \vec{\psi}(n) \end{pmatrix} = A_\lambda \begin{pmatrix} \vec{\psi}(n) \\ \vec{\psi}(n-1) \end{pmatrix}, \tag{1.8}$$

with

$$A_\lambda = \begin{pmatrix} C + \lambda X & -I_l \\ I_l & 0 \end{pmatrix} \quad (l = 2) \tag{1.9}$$

where

$$C = \begin{pmatrix} E & -1 \\ -1 & E \end{pmatrix} \text{ and } X = \begin{pmatrix} -v_{n,1} & 0 \\ 0 & -v_{n,2} \end{pmatrix}.$$

This formulation has the advantage that it generalises to an arbitrary number l of lines.

As in the case of a single line, the eigenvectors are defined up to a multiplicative constant, so only quotients of the components are relevant. These are points of the projective 3-sphere $\mathbb{RP}^{2l-1} = P(\mathbb{R}^{2l})$. The equation for the invariant measure ν_λ^E on $\mathbb{RP}^{2l-1}$ reads:

$$\int_{\mathbb{RP}^{2l-1}} f(x)\nu_\lambda^E(dx) = \int_{\mathbb{RP}^{2l-1}} \mathbb{E}\left(f([A_\lambda x])\right) \nu_\lambda^E(dx),$$

for all $f \in C(\mathbb{RP}^{2l-1})$. X is a random $l \times l$ matrix and $[y]$ denotes the class in $P(\mathbb{R}^{2l})$ containing y. It is convenient to transform A_λ to a more suitable form, J_λ, say, so that the limit $J_0 = \lim_{\lambda \to 0} J_\lambda$, is a the real Jordan form of A_0. Let

$$SA_0S^{-1} = J_0, \tag{1.10}$$

and

$$SA_\lambda S^{-1} = J_\lambda. \tag{1.11}$$

In terms of the image measures

$$\tilde{\nu}_\lambda^E = \nu_\lambda^E \circ \mathcal{S}^{-1}, \tag{1.12}$$

where $\mathcal{S}x = [Sx]$ the invariance equation reads

$$\int_{\mathbb{RP}^{2l-1}} f(x)\, \tilde{\nu}_\lambda^E(dx) = \int_{\mathbb{RP}^{2l-1}} \mathbb{E}\left(f([J_\lambda x])\right) \tilde{\nu}_\lambda^E(dx), \tag{1.13}$$

for all $f \in C(\mathbb{RP}^{2l-1})$. It is convenient to parametrise $\mathbb{RP}^{2l-1}$ by $2l-1$ *angles*. Let Ω be a compact parametrisation space and $t : \mathbb{RP}^{2l-1} \to \Omega$ a parametrisation of $P(\mathbb{R}^{2l})$. The parametrisation for the two particular cases that we consider will be specified later . Defining

$$\sigma_\lambda^E = \tilde{\nu}_\lambda^E \circ t^{-1}, \tag{1.14}$$

the invariance equation becomes

$$\int_\Omega g(\omega)\, \sigma_\lambda^E(d\omega) = \int_\Omega \mathbb{E}\left(g(t[J_\lambda t^{-1}\omega])\right) \sigma_\lambda^E(d\omega), \tag{1.15}$$

or with the notation

$$(\mathcal{T}_\lambda g)(\omega) = \mathbb{E}\left(g(t[J_\lambda t^{-1}\omega])\right), \tag{1.16}$$

$$\int_\Omega g(\omega)\, \sigma_\lambda^E(d\omega) = \int_\Omega (\mathcal{T}_\lambda g)(\omega)\, \sigma_\lambda^E(d\omega). \tag{1.17}$$

Now suppose that σ_λ^E tends to σ_0^E weakly as λ tends to 0 and J_λ tends to J_0. Let

$$(\mathcal{T}_0 g)(\omega) = \left(g(t[J_0 t^{-1}\omega])\right). \tag{1.18}$$

We have by (1.17)

$$\int_\Omega (\mathcal{T}_\lambda g - g)(\omega)\, \sigma_0^E(d\omega) = \int_\Omega (\mathcal{T}_\lambda g - g)(\omega) \left(\sigma_0^E(d\omega) - \sigma_\lambda^E(d\omega)\right). \tag{1.19}$$

Since $\|\mathcal{T}_\lambda g\| \leq \|g\|$,

$$\left|\int_\Omega (\mathcal{T}_\lambda g - g)(\omega) \left(\sigma_0^E(d\omega) - \sigma_\lambda^E(d\omega)\right)\right| \leq 2\|g\|\|\sigma_0 - \sigma_\lambda\| \to 0, \tag{1.20}$$

as $\lambda \to 0$. If $\|\mathcal{T}_0 g - \mathcal{T}_\lambda g\| \to 0$, then

$$\left|\int_\Omega ((\mathcal{T}_0 g) - (\mathcal{T}_\lambda g))(\omega)\, \sigma_0^E(d\omega)\right| \leq \|\mathcal{T}_0 g - \mathcal{T}_\lambda g\| \to 0. \tag{1.21}$$

Therefore

$$\int_\Omega g(\omega)\, \sigma_0^E(d\omega) = \int_\Omega (\mathcal{T}_0 g)(\omega)\, \sigma_0^E(d\omega). \tag{1.22}$$

This invariance equation together with ergodicity is enough in many cases to determine σ_0^E. For the other cases we need the following result. We have, again by (1.17), for any positive integer q,

$$\lambda^{-2}\int_\Omega (\mathcal{T}_\lambda^q g - g)(\omega)\,\sigma_0^E(d\omega) = \lambda^{-2}\int_\Omega (\mathcal{T}_\lambda^q g - g)(\omega)\left(\sigma_0^E(d\omega) - \sigma_\lambda^E(d\omega)\right). \tag{1.23}$$

If $\lambda^{-2}\|\mathcal{T}_\lambda^q g - g\|$ is bounded then right hand side of (1.23) tends to 0 as $\lambda \to 0$ and therefore

$$\lim_{\lambda\to 0}\lambda^{-2}\int_\Omega (\mathcal{T}_\lambda^q g - g)(\omega)\,\sigma_0^E(d\omega) = 0. \tag{1.24}$$

If in addition $\lambda^{-2}(\mathcal{T}_\lambda^q g - g)$ converges pointwise as $\lambda \to 0$ to a function $F_{q,g} \in C(\Omega)$, then

$$\int_\Omega F_{q,g}(\omega)\,\sigma_0^E(d\omega) = 0. \tag{1.25}$$

To be able to exploit (1.23) we shall need the following iteration result.

2 Iteration Formula

In this section we compute the lowest order terms in the expansion of a product of m independent random matrices of the form (1.9). Let C be an $l \times l$ matrix which can be written as $2\cos G$ where G is an $l \times l$ matrix. Let

$$\tau(x,r) = \frac{\sin rx}{\sin x} \tag{2.1}$$

and $T(r) = \tau(G,r)$. Note that

$$T(r) = 2\cos G\, T(r-1) - T(r-2) = 2T(r-1)\cos G - T(r-2), \tag{2.2}$$

Let $A_\lambda^{(n)}$ be a $2l \times 2l$ matrix defined by

$$A_\lambda^{(n)} = \begin{pmatrix} C + \lambda X_n & -I_l \\ I_l & 0 \end{pmatrix} = \begin{pmatrix} 2\cos G + \lambda X_n & -I_l \\ I_l & 0 \end{pmatrix} \tag{2.3}$$

where $X_1, X_2, \ldots$ are independent random $l \times l$ matrices with mean zero and let

$$B(m) = \Pi_{n=1}^m A_\lambda^{(n)}. \tag{2.4}$$

Then

$$B(m) = B_0(m) + \lambda B_1(m) + \lambda^2 B_2(m)(X_1,\ldots,X_m) + \mathrm{O}(\lambda^3), \tag{2.5}$$

where

$$B_0(m) = \begin{pmatrix} T(m+1) & -T(m) \\ T(m) & -T(m-1) \end{pmatrix}, \tag{2.6}$$

$$B_1(m) = \frac{1}{2}\sum_{n=1}^m \begin{pmatrix} T(n) & T(n) \\ T(n-1) & T(n-1) \end{pmatrix}\begin{pmatrix} X_n & 0 \\ 0 & X_n \end{pmatrix}\begin{pmatrix} T(m-n+1) & -T(m-n) \\ T(m-n+1) & -T(m-n) \end{pmatrix}, \tag{2.7}$$

and $\mathbb{E}(B_2(m)) = 0$.

The proof is by induction. We want to show that

$$B(m)_{11} = T(m+1) + \lambda\sum_{n=1}^m T(n)X_nT(m-n+1) + \lambda^2(B_2(m))_{11}(X_1,\ldots,X_m) + \mathrm{O}(\lambda^3), \tag{2.8}$$

where $\mathbb{E}((B_2(m))_{11}) = 0$. This relation is clearly true for $m = 1$.

$$\begin{aligned}
B(m+1)_{11} &= B(m)_{11}(2\cos G + \lambda X_{m+1}) + B(m)_{12} \\
&= 2T(m+1)\cos G + \lambda \sum_{n=1}^{m} 2T(n)X_nT(m-n+1)\cos G \\
&\quad +2\lambda^2(B_2(m))_{11}\cos G + \lambda T(m+1)X_{m+1} \\
&\quad\quad +\lambda^2 \sum_{n=1}^{m} T(n)X_nT(m-n+1)X_{m+1} - T(m) \\
&\quad\quad\quad -\lambda \sum_{n=1}^{m} T(n)X_nT(m-n) + \lambda^2(B_2(m))_{12} + \mathrm{O}(\lambda^3) \\
&= (2T(m+1)\cos G - T(m)) \\
&\quad +\lambda \sum_{n=1}^{m} T(n)X_n(2T(m-n+1)\cos G - T(m-n)) \\
&\quad\quad +\lambda T(m+1)X_{m+1} + \lambda^2(B_2(m+1))_{11}(X_1,\dots X_{m+1}) + \mathrm{O}(\lambda^3) \\
&= T(m+2) + \lambda \sum_{n=1}^{m+1} T(n)X_nT((m+1)-n+1) \\
&\quad\quad +\lambda^2(B_2(m+1))_{11}(X_1\dots X_{m+1}) + \mathrm{O}(\lambda^3).
\end{aligned}$$

where

$$\begin{aligned}
(B_2(m+1))_{11}(X_1,\dots X_{m+1}) = 2(B_2(m))_{11}\cos G + \sum_{n=1}^{m} T(n)X_nT(m-n+1)X_{m+1} \\
+(B_2(m))_{12}
\end{aligned}$$

which implies $\mathbb{E}((B_2(m+1))_{11}) = 0$. The other entries of $B(m+1)$ are checked similarly.

3 The case of a single chain ($l = 1$)

In this section we study the case $l = 1$, i.e. a single chain. In this case, the projective space $P(\mathbb{R}^2)$ is homeomorphic to the circle and there is an obvious parametrisation on $\Omega = [0, \pi)$, identifying 0 and π, defined by the map $t : P(\mathbb{R}^2) \to \Omega$ given by

$$\theta = \begin{cases} \cot^{-1}\frac{x_2}{x_1} \in (0,\pi) & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases} \tag{3.1}$$

We put $C(\Omega) = \{f \mid f \in C([0,\pi]), f(0) = f(\pi)\}$. Recall that $E \in [-2, 2]$ so that we can write $E = 2\cos\alpha$ with $\alpha \in [0, \pi)]$ and

$$A_\lambda = \begin{pmatrix} 2\cos\alpha + \lambda X & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

We first consider the case $E \neq \pm 2$. Then the real Jordan form of A_0 is R_α, the rotation by α:

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.3}$$

We have

$$J_0 = SA_0S^{-1} = R_\alpha \tag{3.4}$$

where

$$S = \begin{pmatrix} \sin\alpha & 0 \\ \cos\alpha & -1 \end{pmatrix}. \tag{3.5}$$

As a result

$$(\mathcal{T}_0 g)(\theta) = g((\theta - \alpha) \bmod \pi). \tag{3.6}$$

If $g \in C(\Omega)$ has bounded first derivative, it follows from (6.14) that $\|\mathcal{T}_0 g - \mathcal{T}_\lambda g\| \to 0$, and therefore for such g the invariance equation (1.22) for σ_0^E holds. If α is not a rational multiple of π, the invariance equation (1.22) and ergodicity imply that σ_0^E is the uniform measure on $[0,\pi)$. If $\alpha = p\pi/q$ is a rational multiple of π, we use the fact that $\mathcal{T}_0^q$ is the identity map, $\mathcal{I}$. If the random variables X_n are symmetric then $\left(\frac{\partial}{\partial\lambda}\mathcal{T}_\lambda^q g\right)_{\lambda=0} = 0$. Therefore, if the first three derivatives of g are bounded, (6.14) of Appendix 2 gives

$$\lim_{\lambda\to 0}\left\|\lambda^{-2}(\mathcal{T}_\lambda^q g - g) - \left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^q g\right)_{\lambda=0}\right\| = 0. \tag{3.7}$$

If $\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^q g\right)_{\lambda=0}$ is continuous, equations (1.24) and (1.25) then yield

$$\int_\Omega \left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^q g\right)_{\lambda=0}(\theta)\,\sigma_0^E(d\theta) = 0. \tag{3.8}$$

We now calculate $\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^q g\right)_{\lambda=0}$ with $g(\theta) = e^{2in\theta}$. Recall that

$$(\mathcal{T}_\lambda^q g)(\theta) = \mathbb{E}\left(g(t[\Pi_{n=1}^q J_\lambda^{(n)} t^{-1}\theta])\right), \tag{3.9}$$

where $J_\lambda^{(n)} = S A_\lambda^{(n)} S^{-1}$. Hence

$$((\mathcal{T}_\lambda^q)\, g)(\theta) = \mathbb{E}\left(g(t[S\left(\Pi_{n=1}^q A_\lambda^{(n)}\right) S^{-1} t^{-1}\theta])\right) = \mathbb{E}\left(g(t[SB(q)S^{-1}t^{-1}\theta])\right). \tag{3.10}$$

We have

$$B(q) = B_0(q) + \lambda B_1(q) + \lambda^2 B_2(q)(X_1,\ldots,X_q) + \mathrm{O}(\lambda^3), \tag{3.11}$$

where $B_0(q) = (-1)^p I_2$ and

$$\begin{aligned} B_1(q)_{11} &= -(-1)^p \sum_{n=1}^q \tau(\alpha,n-1)\tau(\alpha,n)X_n, \\ B_1(q)_{12} &= (-1)^p \sum_{n=1}^q \tau(\alpha,n)^2 X_n, \\ B_1(q)_{21} &= -(-1)^p \sum_{n=1}^q \tau(\alpha,n-1)^2 X_n, \\ B_1(q)_{22} &= (-1)^p \sum_{n=1}^q \tau(\alpha,n-1)\tau(\alpha,n)X_n. \end{aligned} \tag{3.12}$$

We let

$$X = \sum_{n=1}^q \tau(\alpha,n-1)\tau(\alpha,n)X_n, \tag{3.13}$$

$$Y = \sum_{n=1}^q \tau(\alpha,n)^2 X_n, \tag{3.14}$$

$$Z = \sum_{n=1}^q \tau(\alpha,n-1)^2 X_n. \tag{3.15}$$

Then

$$B_1(q) = (-1)^p \begin{pmatrix} -X & Y \\ -Z & X \end{pmatrix}, \tag{3.16}$$

and if $\alpha \neq \frac{\pi}{2}$,

$$\mathbb{E}(X^2) = \mathbb{E}(YZ) = \frac{(3-2\sin^2\alpha)q}{8\sin^4\alpha}, \tag{3.17}$$

$$\mathbb{E}(Y^2) = \mathbb{E}(Z^2) = \frac{3q}{8\sin^4\alpha}, \tag{3.18}$$

$$\mathbb{E}(XY) = \mathbb{E}(ZX) = \frac{3q\cos\alpha}{8\sin^4\alpha}. \tag{3.19}$$

If $\alpha = \frac{\pi}{2}$ then

$$\mathbb{E}(X^2) = \mathbb{E}(XY) = \mathbb{E}(YZ) = \mathbb{E}(XZ) = 0 \tag{3.20}$$

and

$$\mathbb{E}(Y^2) = \mathbb{E}(Z^2) = 1. \tag{3.21}$$

Let

$$\tilde{B}_1(q) = SB_1(q)S^{-1} = (-1)^p \begin{pmatrix} -Z_1 & -Z_2 \\ Z_3 & Z_1 \end{pmatrix}, \tag{3.22}$$

where $Z_1 = X - Y\cos\alpha$, $Z_2 = Y\sin\alpha$, $Z_3 = (Z + Y\cos^2\alpha - 2X\cos\alpha)/\sin\alpha$.
If $\alpha \neq \frac{\pi}{2}$ then

$$\mathbb{E}(Z_2^2) = \mathbb{E}(Z_3^2) = \frac{3q}{8\sin^2\alpha}, \tag{3.23}$$

$$\mathbb{E}(Z_1^2) = \mathbb{E}(Z_2Z_3) = \frac{q}{8\sin^2\alpha}, \tag{3.24}$$

$$\mathbb{E}(Z_1Z_2) = \mathbb{E}(Z_1Z_3) = 0. \tag{3.25}$$

If $\alpha = \frac{\pi}{2}$ then

$$\mathbb{E}(Z_1^2) = \mathbb{E}(Z_1Z_2) = \mathbb{E}(Z_2Z_3) = \mathbb{E}(Z_3Z_1) = 0 \tag{3.26}$$

and

$$\mathbb{E}(Z_2^2) = \mathbb{E}(Z_3^2) = 1. \tag{3.27}$$

Now

$$\tilde{B}(q) \equiv SB(q)S^{-1} = (-1)^p I_2 + \lambda\tilde{B}_1(q) + \lambda^2\tilde{B}_2(q) + \mathrm{O}(\lambda^3). \tag{3.28}$$

where $\mathbb{E}(\tilde{B}_2(q)) = 0$. If we put

$$x = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

and $x' = \tilde{B}(q)x$, then

$$x'_1 = (-1)^p\{(1-\lambda Z_1)\sin\theta - \lambda Z_2\cos\theta\} + \lambda^2 w_1 + \mathrm{O}(\lambda^3) \tag{3.29}$$

and

$$x'_2 = (-1)^p\{\lambda Z_3\sin\theta + (1+\lambda Z_1)\cos\theta\} + \lambda^2 w_2 + \mathrm{O}(\lambda^3) \tag{3.30}$$

where $\mathbb{E}(w) = 0$. Writing

$$x' = \begin{pmatrix} \sin\theta' \\ \cos\theta' \end{pmatrix},$$

so that $\theta' = t[\tilde{B}(q)t^{-1}\theta]$, we have

$$\tan\theta' = \frac{x'_1}{x'_2} = \tan\theta + \lambda U + \lambda^2 V + \mathrm{O}(\lambda^3), \tag{3.31}$$

where

$$U = -2\tan\theta\ Z_1 - \tan^2\theta\ Z_3 - Z_2 \tag{3.32}$$

$$\begin{aligned} V &= 2\tan\theta\ Z_1^2 + \tan^3\theta\ Z_3^2 + Z_1 Z_2 + \tan\theta\ Z_2 Z_3 + 3\tan^2\theta\ Z_3 Z_1 \\ &\qquad + (-1)^p \sec{}^2\theta(w_1\cos\theta - w_2\sin\theta). \end{aligned} \tag{3.33}$$

We then get

$$\begin{aligned} \exp(2in\theta') &= \left(\frac{1+i\tan\theta'}{1-i\tan\theta'}\right)^n \\ &= \exp(2in\theta)\{1+2i\lambda nU\cos^2\theta - 2in\lambda^2\cos^4\theta(U^2(\tan\theta - in) - V\sec{}^2\theta) + \mathrm{O}(\lambda^3)\}. \end{aligned}$$

Therefore

$$\mathbb{E}\left(\exp(2in\theta')\right) = \exp(2in\theta)\{1 - 2i\lambda^2[n\cos^4\theta(\mathbb{E}(U^2)(\tan\theta - in) - \mathbb{E}(V)\sec{}^2\theta)] + \mathrm{O}(\lambda^3)\}, \tag{3.34}$$

and thus

$$\lim_{\lambda\to 0}\lambda^{-2}\mathbb{E}\left[\exp(2in\theta') - \exp(2in\theta)\right] = \exp(2in\theta)\{A_1 n + A_{11} n^2\}, \tag{3.35}$$

where

$$A_1 = 2i\cos^4\theta(\mathbb{E}(V)\sec{}^2\theta - \mathbb{E}(U^2)\tan\theta), \tag{3.36}$$

$$A_{11} = -2\mathbb{E}(U^2)\cos^4\theta. \tag{3.37}$$

If $\alpha \neq \frac{\pi}{2}$,

$$\mathbb{E}(U^2) = \frac{3q\sec{}^4\theta}{8\sin^2\alpha}, \qquad \mathbb{E}(V) = \frac{3q\tan\theta\sec{}^2\theta}{8\sin^2\alpha}, \tag{3.38}$$

and

$$A_1 = 0, \qquad A_{11} = -\frac{3q}{4\sin^2\alpha}. \tag{3.39}$$

If $\alpha = \frac{\pi}{2}$,

$$\mathbb{E}(U^2) = 1 + \tan^4\theta, \qquad \mathbb{E}(V) = \tan^3\theta, \tag{3.40}$$

and

$$A_1 = 2i(\cos\theta\sin^3\theta - \sin\theta\cos^3\theta) = -\frac{1}{2}i\sin 4\theta, \tag{3.41}$$

$$A_{11} = -2(\sin^4\theta + \cos^4\theta) = -\frac{1}{2}(\cos 4\theta + 3). \tag{3.42}$$

From (3.8) with $g(\theta) = e^{2in\theta}$ we have

$$\int_{[-\frac{\pi}{2},\frac{\pi}{2})} e^{2in\theta}\{A_1(\theta) + nA_{11}(\theta)\}\sigma_0^E(d\theta) = 0 \tag{3.43}$$

for $n \neq 0$. Recall that the set $\{e^{2in\theta} \mid n \in \mathbb{Z}\}$ is total in the space $C(\Omega)$. In the case when $\alpha \neq \frac{\pi}{2}$, (3.43) gives immediately

$$\int_{[0,\pi)} e^{2in\theta}\sigma_0^E(d\theta) = 0 \tag{3.44}$$

for $n \neq 0$, and therefore $\sigma_0^E(d\theta) = \frac{d\theta}{\pi}$. In the case when $\alpha = \frac{\pi}{2}$, that is $E = 0$, if X is a symmetric random variable then σ_0^0 is symmetric about $\frac{\pi}{2}$. It can be seen from the invariance equation that if σ_λ^0 is an invariant measure then so is its reflection about $\frac{\pi}{2}$. By the uniqueness of the invariant measure for $\lambda \neq 0$ it follows that σ_λ^0 is symmetric and therefore so is σ_0^0. We can integrate by parts in (3.43) to get

$$\int_{[0,\pi)} e^{2in\theta} A_1(\theta)\sigma_0^0(d\theta) = -\int_{[0,\pi)} A_1(\theta)\sigma_0^0(d\theta) - 2in\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2in\theta}\int_{[0,\theta)} A_1(\theta')\sigma_0^0(d\theta')d\theta. \tag{3.45}$$

Since σ_0^0 is symmetric about $\frac{\pi}{2}$,

$$\int_{[0,\pi)} A_1(\theta)\sigma_0^0(d\theta) = 0, \tag{3.46}$$

and equation (3.45) gives

$$2i\int_{[0,\pi)} e^{2in\theta}\int_{[0,\theta)} A_1(\theta')\sigma_0^0(d\theta')d\theta = \int_{[0,\pi)} e^{2in\theta} A_{11}(\theta)\sigma_0^0(d\theta). \tag{3.47}$$

Hence

$$A_{11}(\theta)\sigma_0^0(d\theta) = 2i\int_{[0,\theta)} A_1(\theta')\sigma_0^0(d\theta')d\theta + Kd\theta, \tag{3.48}$$

where K is a constant. Since $A_{11}(\theta) \neq 0$, this implies that σ_0^0 is absolutely continuous. If ρ_0 is the density of σ_0^0 then

$$A_{11}(\theta)\rho_0(\theta) = 2i\int_{[0,\theta)} A_1(\theta')\rho_0(\theta')d\theta' + K. \tag{3.49}$$

Thus ρ_0 is differentiable and

$$(\cos 4\theta + 3)\rho_0'(\theta) = 2\sin 4\theta\rho_0(\theta). \tag{3.50}$$

Integrating we get

$$\rho_0(\theta) = C(\cos 4\theta + 3)^{-\frac{1}{2}}. \tag{3.51}$$

This corresponds to the equation (1.6) for $E = 0$.

Now suppose that $E = 2$. The case $E = -2$ is similar. Here the real Jordan form for A_0

$$J_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.52}$$

The matrix S is now given by

$$S_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.53}$$

Note that

$$J_0^q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \tag{3.54}$$

and therefore

$$(\mathcal{T}_0^q g)(\theta) = g(\theta^{(q)}), \tag{3.55}$$

where $\theta^{(q)}$ is given by

$$\cot\theta^{(q)} = \begin{cases} \frac{1}{q} & \text{if } \theta = 0, \\ \frac{\cot\theta}{1+q\cot\theta}, & \text{if } \theta \neq 0, \end{cases} \tag{3.56}$$

It follows that $\theta^{(q)} \to \frac{\pi}{2}$ as $q \to \infty$.

We now have
$$\int_\Omega g(\theta)\,\sigma_0^E(d\theta) = \lim_{q\to\infty}\int_\Omega (\mathcal{T}_0^q g)(\theta)\,\sigma_0^E(d\theta). \tag{3.57}$$
Thus we have, for $n \in \mathbb{Z}$,
$$\int_{\Omega_0} e^{2in\theta}\,\sigma_0^1(d\theta) = \int_{\Omega\cap\{\theta=\frac{\pi}{2}\}} e^{2in_2\theta_2}\,\sigma_0^1(d\theta). \tag{3.58}$$
Therefore σ_0^1 is concentrated on $\Omega\cap\{\theta=\frac{\pi}{2}\}$, i.e. $\sigma_0^1 = \delta_{\pi/2}$.

To investigate whether there is an anomaly at $E = 2$ we need to transform the invariant measure $d\theta$ to the coordinates given by the matrix S_2. Calling the new angle coordinate θ', the transformation is given by
$$\begin{pmatrix}\sin\theta'\\ \cos\theta'\end{pmatrix} = S_2S^{-1}\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix} \tag{3.59}$$
and
$$S_2S^{-1} = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\operatorname{cosec}\alpha & 0\\ \cot\alpha & -1\end{pmatrix} = \begin{pmatrix}\cot\alpha & -1\\ \frac{1-\cos\alpha}{\sin\alpha} & 1\end{pmatrix}. \tag{3.60}$$
Hence
$$\cot\theta' = \frac{\sin\alpha\cot\theta' + 1 - \cos\alpha}{\cos\alpha - \sin\alpha\cot\theta'} \tag{3.61}$$
and
$$d\theta = \frac{d\theta'}{\sin^2\theta'}\,\frac{\sin\alpha}{\cot^2\theta' + 2(1-\cos\alpha)(1+\cot\theta')}. \tag{3.62}$$
As α tends to 0, i.e. $E \to 2$, this measure tends to $\delta_{\pi/2}$, so there is no (zeroth-order) anomaly at $E = 2$.

4 The case of two coupled chains ($l = 2$)

4.1 Parametrisation

In the case of two coupled chains ($l = 2$), the matrix C in (1.9) is given by
$$C = \begin{pmatrix}E & -1\\ -1 & E\end{pmatrix}. \tag{4.1}$$
If
$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{4.2}$$
then $C = UDU^*$ where
$$D = \begin{pmatrix}E+1 & 0\\ 0 & E-1\end{pmatrix}. \tag{4.3}$$
We can write $D = 2\cos D_0$ with
$$D_0 = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \tag{4.4}$$
where α and β are defined by $2\cos\alpha = E+1$ and $2\cos\beta = E-1$. Note that α and β are not always real. It follows that $G = UD_0U^*$ and $T(r) = U\tau(D_0)U^*$. Thus
$$T(r) = \frac{1}{2}\begin{pmatrix}\tau(\alpha,r)+\tau(\beta,r) & -\tau(\alpha,r)+\tau(\beta,r)\\ -\tau(\alpha,r)+\tau(\beta,r) & \tau(\alpha,r)+\tau(\beta,r)\end{pmatrix}. \tag{4.5}$$

The real Jordan form of A_0 is always of the form

$$J_0 = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}. \tag{4.6}$$

It is therefore convenient to parametrise the projective space $\mathbb{RP}^3$ so that the 1-2 plane and the 3-4 plane have the usual parametrisation.

We map the projective space $\mathbb{RP}^3$ onto the set $\Omega = \Omega_{(0,\frac{\pi}{2})} \cup \Omega_0 \cup \Omega_{\frac{\pi}{2}}$ where

$$\Omega_{(0,\frac{\pi}{2})} = [0,2\pi) \times [0,\pi) \times (0,\frac{\pi}{2}), \quad \Omega_0 = [0,\pi) \times \{0\}, \quad \Omega_{\frac{\pi}{2}} = [0,\pi) \times \{\frac{\pi}{2}\} \tag{4.7}$$

by the mapping $t : \mathbb{RP}^3 \to \Omega$ defined as follows.
If $x_1^2 + x_2^2 \neq 0$ and $x_3^2 + x_4^2 \neq 0$,

$$t(x) = (\theta_1, \theta_2, \theta_3) \in \Omega_{(0,\frac{\pi}{2})}, \tag{4.8}$$

where

$$\theta_1 = \begin{cases} \cot^{-1}\frac{x_2}{x_1} \in (0,\pi) & \text{if } x_1 > 0, \\ \cot^{-1}\frac{x_2}{x_1} + \pi \in (\pi, 2\pi) & \text{if } x_1 < 0, \\ 0 & \text{if } x_1 = 0 \text{ and } x_2 > 0, \\ \pi & \text{if } x_1 = 0 \text{ and } x_2 < 0. \end{cases} \tag{4.9}$$

$$\theta_2 = \begin{cases} \cot^{-1}\frac{x_4}{x_3} \in (0,\pi) & \text{if } x_3 \neq 0, \\ 0 & \text{if } x_3 = 0. \end{cases} \tag{4.10}$$

$$\theta_3 = \cot^{-1}\sqrt{\frac{x_3^2 + x_4^2}{x_1^2 + x_2^2}} \in (0, \frac{\pi}{2}). \tag{4.11}$$

If $x_1^2 + x_2^2 = 0$,

$$t(x) = (\theta_2, 0) \in \Omega_0, \tag{4.12}$$

where

$$\theta_2 = \begin{cases} \cot^{-1}\frac{x_4}{x_3} \in (0,\pi) & \text{if } x_3 \neq 0, \\ 0 & \text{if } x_3 = 0. \end{cases} \tag{4.13}$$

If $x_3^2 + x_4^2 = 0$,

$$t(x) = (\theta_1, \frac{\pi}{2}) \in \Omega_{\frac{\pi}{2}}, \tag{4.14}$$

where

$$\theta_1 = \begin{cases} \cot^{-1}\frac{x_2}{x_1} \in (0,\pi) & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases} \tag{4.15}$$

We give the induced topology on Ω by describing the continuous functions $C(\Omega)$ on Ω. For $f : \Omega \to \mathbb{C}$ define $f_{(0,\frac{\pi}{2})} = f|\,\Omega_{(0,\frac{\pi}{2})}$, $f_0 = f|\,\Omega_0$ and $f_{\frac{\pi}{2}} = f|\,\Omega_{\frac{\pi}{2}}$. Now, f is in $C(\Omega)$ if $f_{(0,\frac{\pi}{2})}$, f_0 and $f_{\frac{\pi}{2}}$ are continuous and

$$\begin{aligned}
\lim_{\theta_2 \to \pi} f_0(\theta_2, 0) &= f_0(0,0), & (4.16)\\
\lim_{\theta_1 \to \pi} f_{\frac{\pi}{2}}(\theta_1, \frac{\pi}{2}) &= f_{\frac{\pi}{2}}(0, \frac{\pi}{2}), & (4.17)\\
\lim_{\theta_1 \to 2\pi} f_{(0,\frac{\pi}{2})}(\theta_1, \theta_2, \theta_3) &= f_{(0,\frac{\pi}{2})}(0, \theta_2, \theta_3), & (4.18)\\
\lim_{\theta_3 \to 0} f_{(0,\frac{\pi}{2})}(\theta_1, \theta_2, \theta_3) &= f_0(\theta_2, 0), & (4.19)\\
\lim_{\theta_3 \to \frac{\pi}{2}} f_{(0,\frac{\pi}{2})}(\theta_1, \theta_2, \theta_3) &= f_{\frac{\pi}{2}}(\theta_1 \bmod \pi, \frac{\pi}{2}), & (4.20)\\
\lim_{\theta_2 \to \pi} f_{(0,\frac{\pi}{2})}(\theta_1, \theta_2, \theta_3) &= f_{(0,\frac{\pi}{2})}((\theta_1 + \pi) \bmod 2\pi, 0, \theta_3). & (4.21)
\end{aligned}$$

Suppose $f \in C(\Omega)$. Then we can write $f = f^{(1)} + f^{(2)} + f^{(3)}$ where $f^{(1)}$, $f^{(2)}$, and $f^{(3)} \in C(\Omega)$ are defined by

$$\begin{aligned}
f^{(1)}_{(0,\frac{\pi}{2})}(\theta_1,\theta_2,\theta_3) &= f_{(0,\frac{\pi}{2})}(\theta_1,\theta_2,\theta_3) - f_0(\theta_2,0)\cos\theta_3 - f_{\frac{\pi}{2}}(\theta_1 \bmod \pi, \tfrac{\pi}{2})\sin\theta_3, && (4.22)\\
f^{(1)}_0(\theta_2,0) &= 0, && (4.23)\\
f^{(1)}_{\frac{\pi}{2}}(\theta_1,\tfrac{\pi}{2}) &= 0, && (4.24)\\
& && (4.25)\\
f^{(2)}_{(0,\frac{\pi}{2})}(\theta_1,\theta_2,\theta_3) &= f_{\frac{\pi}{2}}(\theta_1 \bmod \pi, \tfrac{\pi}{2})\sin\theta_3, && (4.26)\\
f^{(2)}_0(\theta_2,0) &= 0, && (4.27)\\
f^{(2)}_{\frac{\pi}{2}}(\theta_1,\tfrac{\pi}{2}) &= f_{\frac{\pi}{2}}(\theta_1,\tfrac{\pi}{2}), && (4.28)\\
& && (4.29)\\
f^{(3)}_{(0,\frac{\pi}{2})}(\theta_1,\theta_2,\theta_3) &= f_0(\theta_2,0)\cos\theta_3, && (4.30)\\
f^{(3)}_0(\theta_2,0) &= f_0(\theta_2,0), && (4.31)\\
f^{(3)}_{\frac{\pi}{2}}(\theta_1,\tfrac{\pi}{2}) &= 0. && (4.32)
\end{aligned}$$

It is clear from the above decomposition that the union of the following three sets is total in $C(\Omega)$:

$$\{e^{i(n_1\theta_1+n_2\theta_2)}\sin 2n_3\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} \mid n_1,n_2 \in \mathbb{Z}, n_3 \in \mathbb{N}, n_1+n_2 \text{ even}\}, \tag{4.33}$$

$$\{e^{2in_1\theta_1}\sin\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_1\theta_1}\,\mathbf{1}_{\Omega_{\frac{\pi}{2}}} \mid n_1 \in \mathbb{Z}\}, \tag{4.34}$$

$$\{e^{2in_2\theta_2}\cos\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_2\theta_2}\,\mathbf{1}_{\Omega_0} \mid n_2 \in \mathbb{Z}\}. \tag{4.35}$$

In fact, it will be more convenient to use as a total set the union of the following three sets with $r \in \mathbb{N}_0$:

$$\{e^{i(n_1\theta_1+n_2\theta_2)}\cos 2n_3\theta_3\,\sin^{2(r+1)}2\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} \mid n_1,n_2 \in \mathbb{Z}, n_3 \in \mathbb{N}, n_1+n_2 \text{ even}\}, \tag{4.36}$$

$$\{e^{2in_1\theta_1}\sin^{2(r+1)}\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_1\theta_1}\,\mathbf{1}_{\Omega_{\frac{\pi}{2}}} \mid n_1 \in \mathbb{Z}\}, \tag{4.37}$$

$$\{e^{2in_2\theta_2}\cos^{2(r+1)}\theta_3\,\mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_2\theta_2}\,\mathbf{1}_{\Omega_0} \mid n_2 \in \mathbb{Z}\}. \tag{4.38}$$

This is because, as we shall see in Appendix 2, if g is an element of this total set then it satisfies

$$\lim_{\lambda\to 0}\lambda^{-r}\left\|\mathcal{T}^q_\lambda g - \sum_{k=0}^{r}\frac{\lambda^k}{k!}\left(\frac{\partial^k}{\partial\lambda^k}\mathcal{T}^q_\lambda g\right)_{\lambda=0}\right\| = 0. \tag{4.39}$$

4.2 General Scheme

We shall assume that the X_n's are diagonal, that is,

$$X_n = \begin{pmatrix} X^{(1)}_n & 0 \\ 0 & X^{(2)}_n \end{pmatrix}. \tag{4.40}$$

Let

$$\tilde{B}(m) = SB(m)S^{-1}. \tag{4.41}$$

Then

$$\tilde{B}(m) = \tilde{B}_0(m) + \lambda\tilde{B}_1(m) + \lambda^2\tilde{B}_2(m)(X_1, \ldots, X_m) + \mathrm{O}(\lambda^3), \tag{4.42}$$

where

$$\tilde{B}_0(m) = J_0^m, \tag{4.43}$$

$$\tilde{B}_1(m) = SB_1(m)S^{-1}, \tag{4.44}$$

and $\mathbb{E}(\tilde{B}_2(m)) = 0$. $\tilde{B}_1(m)$ can be expressed in the form

$$\tilde{B}_1(m) = \sum_{n=1}^{m} Y_n^- C_n(m) + \sum_{n=1}^{m} Y_n^+ D_n(m), \tag{4.45}$$

where $Y_n^{\pm} = \frac{1}{2}(X_n^{(1)} \pm X_n^{(2)})$.

Let

$$(\mathcal{T}_0^m g)(\theta_1, \theta_2, \theta_3) = g(\theta_1^{(m)}, \theta_2^{(m)}, \theta_3^{(m)}), \tag{4.46}$$

$$x = \begin{pmatrix} \sin\theta_1 \sin\theta_3 \\ \cos\theta_1 \sin\theta_3 \\ \sin\theta_2 \cos\theta_3 \\ \cos\theta_2 \cos\theta_3 \end{pmatrix} \tag{4.47}$$

and $x' = \tilde{B}(m)x$. Then

$$x' = \begin{pmatrix} \sin\theta_1^{(m)} \sin\theta_3^{(m)} \\ \cos\theta_1^{(m)} \sin\theta_3^{(m)} \\ \sin\theta_2^{(m)} \cos\theta_3^{(m)} \\ \cos\theta_2^{(m)} \cos\theta_3^{(m)} \end{pmatrix} + \lambda y + \lambda^2 w + \mathrm{O}(\lambda^3), \tag{4.48}$$

where $\mathbb{E}(w) = 0$ and

$$y_i = \sum_{n=1}^{m} Y_n^- \langle C_n(m)^T e_i, x\rangle + \sum_{n=1}^{m} Y_n^+ \langle D_n(m)^T e_i, x\rangle. \tag{4.49}$$

$\mathbb{E}(y) = 0$ and

$$\mathbb{E}(y_i y_j) = \frac{1}{2}\left(\sum_{n=1}^{m}\langle C_n(m)^T e_i, x\rangle\langle C_n(m)^T e_j, x\rangle + \sum_{n=1}^{m}\langle D_n(m)^T e_i, x\rangle\langle D_n(m)^T e_j, x\rangle\right), \tag{4.50}$$

where $\{e_1, e_2, e_3, e_4\}$ is the usual orthonormal basis in $\mathbb{R}^4$. Writing

$$x' = \begin{pmatrix} \sin\theta_1' \sin\theta_3' \\ \cos\theta_1' \sin\theta_3' \\ \sin\theta_2' \cos\theta_3' \\ \cos\theta_2' \cos\theta_3' \end{pmatrix}, \tag{4.51}$$

we have

$$\tan\theta_1' = \frac{x_1'}{x_2'} = \tan\theta_1^{(m)} + \lambda U_1 + \lambda^2 V_1 + \mathrm{O}(\lambda^3), \tag{4.52}$$

where

$$U_1 = \frac{y_1\cos\theta_1^{(m)} - y_2\sin\theta_1^{(m)}}{\cos^2\theta_1^{(m)}\sin\theta_3^{(m)}} \tag{4.53}$$

and

$$V_1 = -\frac{y_1 y_2 \cos\theta_1^{(m)} - y_2^2 \sin\theta_1^{(m)}}{\cos^3\theta_1^{(m)} \sin^2\theta_3^{(m)}} + W_1, \tag{4.54}$$

with $\mathbb{E}(W_1) = 0$. Next we have

$$\tan\theta_2' = \frac{x_3'}{x_4'} = \tan\theta_2^{(m)} + \lambda U_2 + \lambda^2 V_2 + \mathrm{O}(\lambda^3), \tag{4.55}$$

where

$$U_2 = \frac{y_3 \cos\theta_2^{(m)} - y_4 \sin\theta_2^{(m)}}{\cos^2\theta_2^{(m)} \cos\theta_3^{(m)}} \tag{4.56}$$

and

$$V_2 = -\frac{y_3 y_4 \cos\theta_2^{(m)} - y_4^2 \sin\theta_2^{(m)}}{\cos^3\theta_2^{(m)} \cos^2\theta_3^{(m)}} + W_2, \tag{4.57}$$

with $\mathbb{E}(W_2) = 0$. Thirdly,

$$\tan\theta_3' = \left(\frac{(x_1')^2 + (x_2')^2}{(x_3')^2 + (x_4')^2}\right)^{\frac{1}{2}} = \tan\theta_3^{(m)} + \lambda U_3 + \lambda^2 V_3 + \mathrm{O}(\lambda^3), \tag{4.58}$$

where

$$U_3 = \frac{\cos\theta_3^{(m)}(y_1 \sin\theta_1^{(m)} + y_2 \cos\theta_1^{(m)}) - \sin\theta_3^{(m)}\,(y_3 \sin\theta_2^{(m)} + y_4 \cos\theta_2^{(m)})}{\cos^2\theta_3^{(m)}} \tag{4.59}$$

and

$$\begin{aligned} V_3 = {} & \frac{y_1^2 \cos^2\theta_1^{(m)} + y_2^2 \sin^2\theta_1^{(m)} - 2y_1 y_2 \sin\theta_1^{(m)} \cos\theta_1^{(m)}}{2\sin\theta_3^{(m)} \cos\theta_3^{(m)}} \\ & + \sin\theta_3^{(m)} \frac{y_3^2(3\sin^2\theta_2^{(m)} - 1) + y_4^2(3\cos^2\theta_2^{(m)} - 1) + 6y_3 y_4 \sin\theta_2^{(m)} \cos\theta_2^{(m)}}{2\cos^3\theta_3} \\ & - \frac{y_2 y_4 \cos\theta_1^{(m)} \cos\theta_2^{(m)} + y_1 y_3 \sin\theta_1^{(m)} \sin\theta_2^{(m)} + y_2 y_3 \cos\theta_1^{(m)} \sin\theta_2^{(m)} + y_1 y_4 \cos\theta_2^{(m)} \sin\theta_1^{(m)}}{\cos^2\theta_3^{(m)}} \\ & + W_3, \end{aligned} \tag{4.60}$$

where $\mathbb{E}(W_3) = 0$. For $k = 1, 2, 3$, therefore,

$$\begin{aligned} \exp(in_k\theta_k') &= \left(\frac{1 + i\tan\theta_k'}{1 - i\tan\theta_k'}\right)^{\frac{1}{2}n_k} \\ &= \exp(in_k\theta_k^{(m)})\Big\{1 + i\lambda n_k U_k \cos^2\theta_k^{(m)} + in_k\lambda^2 V_k \cos^2\theta_k^{(m)} \\ &\qquad - in_k\lambda^2 \cos^4\theta_k^{(m)} U_k^2 \left(\tan\theta_k^{(m)} - \frac{1}{2}in_k\right)\Big\} + \mathrm{O}(\lambda^3). \end{aligned}$$

Hence

$$\begin{aligned} \exp(i(n_1\theta_1' + n_2\theta_2' + n_3\theta_3')) &= \exp(i(n_1\theta_1^{(m)} + n_2\theta_2^{(m)} + n_3\theta_3^{(m)})) \\ &\times\Big\{1 + i\lambda[n_1 U_1 \cos^2\theta_1^{(m)} + n_2 U_2 \cos^2\theta_2^{(m)} + n_3 U_3 \cos^2\theta_3^{(m)}] \\ &+\lambda^2[B_1 n_1 + B_2 n_2 + B_3 n_3 + B_{11} n_1^2 + B_{22} n_2^2 + B_{33} n_3^2 \\ &+B_{12} n_1 n_2 + B_{23} n_2 n_3 + B_{31} n_3 n_1]\Big\} + \mathrm{O}(\lambda^3). \end{aligned} \tag{4.61}$$

where

$$B_k = i\left(V_k \cos^2\theta_k^{(m)} - U_k^2 \tan\theta_k^{(m)} \cos^4\theta_k^{(m)}\right), \tag{4.62}$$

$$B_{kk} = -\frac{1}{2} U_k^2 \cos^4\theta_k^{(m)}, \tag{4.63}$$

and for $k \neq l$,

$$B_{kl} = -U_k U_l \cos^2\theta_k^{(m)} \cos^2\theta_l^{(m)}. \tag{4.64}$$

Taking expectations we get

$$\begin{aligned}\mathbb{E}\left(\exp(i(n_1\theta_1' + n_2\theta_2' + n_3\theta_3'))\right) &= \exp(i(n_1\theta_1^{(m)} + n_2\theta_2^{(m)} + n_3\theta_3^{(m)})) \\ &\times\Big\{1 + \lambda^2[A_1n_1 + A_2n_2 + A_3n_3 + A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 \\ &+A_{12}n_1n_2 + A_{23}n_2n_3 + A_{31}n_3n_1]\Big\} + \mathrm{O}(\lambda^3),\end{aligned} \tag{4.65}$$

where $A_k = \mathbb{E}(B_k)$ and $A_{kl} = \mathbb{E}(B_{kl})$. The right-hand side of this equation can be written as:

$$\begin{aligned}&\Big\{ - iA_1\frac{\partial}{\partial\theta_1^{(m)}} - iA_2\frac{\partial}{\partial\theta_2^{(m)}} - iA_3\frac{\partial}{\partial\theta_3^{(m)}} - A_{11}\frac{\partial^2}{\partial\theta_1^{(m)2}} - A_{22}\frac{\partial^2}{\partial\theta_2^{(m)2}} - A_{33}\frac{\partial^2}{\partial\theta_3^{(m)2}} \\ &-A_{12}\frac{\partial^2}{\partial\theta_1^{(m)}\partial\theta_2^{(m)}} - A_{23}\frac{\partial^2}{\partial\theta_2^{(m)}\partial\theta_3^{(m)}} - A_{31}\frac{\partial^2}{\partial\theta_3^{(m)}\partial\theta_1^{(m)}}\Big\}\exp(i(n_1\theta_1^{(m)} + n_2\theta_2^{(m)} + n_3\theta_3^{(m)})) \\ &+\mathrm{O}(\lambda^3).\end{aligned} \tag{4.66}$$

4.3 The case $E \in (-1, 1)$

If $-1 < E < 1$ we can choose $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in (\frac{\pi}{2}, \pi)$ satisfying $2\cos\alpha = E + 1$ and $2\cos\beta = E - 1$. The real Jordan form of A_0 is

$$J_0 = \begin{pmatrix} R_\alpha & 0 \\ 0 & R_\beta \end{pmatrix} \tag{4.67}$$

where

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{4.68}$$

$$S = \begin{pmatrix} 1 & -1 & -\cos\alpha & \cos\alpha \\ 0 & 0 & \sin\alpha & -\sin\alpha \\ -\cos\beta & -\cos\beta & 1 & 1 \\ -\sin\beta & -\sin\beta & 0 & 0 \end{pmatrix}, \tag{4.69}$$

and

$$S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \cot\alpha & 0 & -\operatorname{cosec}\beta \\ -1 & -\cot\alpha & 0 & -\operatorname{cosec}\beta \\ 0 & \operatorname{cosec}\alpha & 1 & -\cot\beta \\ 0 & -\operatorname{cosec}\alpha & 1 & -\cot\beta \end{pmatrix}. \tag{4.70}$$

Note that

$$(\mathcal{T}_0 g)(\theta_1, \theta_2, \theta_3) = g((\theta_1 - \alpha) \bmod 2\pi, (\theta_2 - \beta) \bmod \pi, \theta_3) \tag{4.71}$$

and therefore

$$\theta_1^{(m)} = (\theta_1 - m\alpha) \bmod 2\pi, \quad \theta_2^{(m)} = (\theta_2 - m\beta) \bmod \pi, \quad \theta_3^{(m)} = \theta_3. \tag{4.72}$$

Consider the total set

$$\{f_{n_1,n_2,n_3} |\ n_1, n_2 \in \mathbb{Z}, n_3 \in \mathbb{N}, n_1 + n_2 \text{ even}\} \cup \{g_{n_1} |\ n_1 \in \mathbb{Z}\} \cup \{h_{n_2} |\ n_2 \in \mathbb{Z}\}, \tag{4.73}$$

where

$$f_{n_1,n_2,n_3}(\theta_1, \theta_2, \theta_3) = e^{i(n_1\theta_1+n_2\theta_2)} \cos 2n_3\theta_3 \sin^2 2\theta_3 \, \mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}}, \tag{4.74}$$

$$g_{n_1}(\theta_1, \theta_3) = e^{2in_1\theta_1} \sin^2 \theta_3 \, \mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_1\theta_1} \, \mathbf{1}_{\Omega_{\frac{\pi}{2}}} \tag{4.75}$$

and

$$h_{n_2}(\theta_2, \theta_3) = e^{2in_2\theta_2} \cos^2 \theta_3 \, \mathbf{1}_{\Omega_{(0,\frac{\pi}{2})}} + e^{2in_2\theta_2} \, \mathbf{1}_{\Omega_0}. \tag{4.76}$$

If g is in this total set then it satisfies (4.39) with $r = 0$, that is,

$$\lim_{\lambda\to 0} \|\mathcal{T}_\lambda g - \mathcal{T}_0 g\| = 0, \tag{4.77}$$

and therefore

$$\mathcal{T}_0^* \sigma_0^E = \sigma_0^E, \tag{4.78}$$

that is, σ_0^E is invariant under rotations of θ_1 and θ_2 by α and β respectively. This is all we can say unless one of α/π or β/π are irrational. Consider the case when both α/π and β/π are irrational. Because of the relation between α and β, $(n_1\alpha + n_2\beta)/\pi$ is also irrational for any $n_1, n_2 \in \mathbb{Z}$. The standard ergodic argument then shows that σ_0^E is Lebesgue with respect to θ_1 and θ_2, that is, on $\Omega_{(0,\frac{\pi}{2})}$, $\sigma_0^E(d\theta_1, d\theta_2, d\theta_3) = d\theta_1 d\theta_2 \, \tilde{\sigma}_0^E(d\theta_3)$, on $\Omega_{\frac{\pi}{2}}$, $\sigma_0^E(d\theta_1) = \delta_{\frac{\pi}{2}} d\theta_1$ and on Ω_0, $\sigma_0^E(d\theta_2) = \delta_0 d\theta_2$.

The ergodic argument goes like this:

$$\langle f_{n_1,n_2,n_3}, \sigma_0^E \rangle = e^{i(n_1\alpha+n_2\beta)} \langle f_{n_1,n_2,n_3}, \sigma_0^E \rangle, \quad \langle g_{n_1}, \sigma_0^E \rangle = e^{in_1\alpha} \langle g_{n_1}, \sigma_0^E \rangle, \tag{4.79}$$

and

$$\langle h_{n_2}, \sigma_0^E \rangle = e^{in_2\beta} \langle h_{n_2}, \sigma_0^E \rangle. \tag{4.80}$$

Therefore $\langle f_{n_1,n_2,n_3}, \sigma_0^E \rangle = 0$ if $n_1 \neq 0$ and $n_2 \neq 0$, $\langle g_{n_1,n_3}, \sigma_0^E \rangle = 0$ if $n_1 \neq 0$ and $\langle h_{n_2,n_3}, \sigma_0^E \rangle = 0$ if $n_2 \neq 0$. Define

$$\tilde{\sigma}_0^E(d\theta_3) = \frac{1}{2\pi^2} \int_{[0,2\pi)\times[0,\pi)} \sigma_0^E(d\theta_1, d\theta_2, d\theta_3) \tag{4.81}$$

and

$$\delta_{\frac{\pi}{2}} = \frac{1}{\pi} \sigma_0^E(\Omega_{\frac{\pi}{2}}) \quad \text{and} \quad \delta_0 = \frac{1}{\pi} \sigma_0^E(\Omega_0). \tag{4.82}$$

If on $\Omega_{(0,\frac{\pi}{2})}$, $\hat{\sigma}_0^E(d\theta_1, d\theta_2, d\theta_3) = d\theta_1 d\theta_2 \, \tilde{\sigma}_0^E(d\theta_3)$, on $\Omega_{\frac{\pi}{2}}$, $\hat{\sigma}_0^E(d\theta_1) = \delta_{\frac{\pi}{2}} d\theta_1$ and on Ω_0, $\hat{\sigma}_0^E(d\theta_2) = \delta_0 d\theta_2$, then

$$\langle f_{n_1,n_2,n_3}, \hat{\sigma}_0^E \rangle = \langle f_{n_1,n_2,n_3}, \sigma_0^E \rangle, \quad \langle g_{n_1}, \hat{\sigma}_0^E \rangle = \langle g_{n_1}, \sigma_0^E \rangle, \tag{4.83}$$

and

$$\langle h_{n_2}, \hat{\sigma}_0^E \rangle = \langle h_{n_2}, \sigma_0^E \rangle. \tag{4.84}$$

Therefore $\hat{\sigma}_0^E = \sigma_0^E$.

A similar argument applies if only one of α/π or β/π is irrational. Suppose for example, that α/π is irrational and $\beta/\pi = p/q$ where p and q are integers. Then, replacing $\mathcal{T}_0$ with $\mathcal{T}_0^q$ in the above argument, we can see that σ_0^E is Lebesgue with respect to θ_1, that is, on $\Omega_{(0,\frac{\pi}{2})}$, $\sigma_0^E(d\theta_1, d\theta_2, d\theta_3) = d\theta_1 \tilde{\sigma}_0^E(d\theta_2, d\theta_3)$ and on $\Omega_{\frac{\pi}{2}}$, $\sigma_0^E(d\theta_1) = \delta_{\frac{\pi}{2}} d\theta_1$.

In this case the matrices $C_n(m)$ and $D_n(m)$ in (4.45) are given by

$$C_n(m) = -2\begin{pmatrix} 0 & \begin{array}{l}\frac{1}{\sin\beta}(R_{\frac{\pi}{2}-(n-1)\alpha-(m-n)\beta} \\ \quad +R_{\frac{\pi}{2}-(n-1)\alpha+(m-n)\beta}\sigma_z)\end{array} \\ \begin{array}{l}\frac{1}{\sin\alpha}(R_{\frac{\pi}{2}-n\beta-(m-n+1)\alpha} \\ \quad +R_{\frac{\pi}{2}-n\beta+(m-n+1)\alpha}\sigma_z)\end{array} & 0 \end{pmatrix} \tag{4.85}$$

and

$$D_n(m) = 2\begin{pmatrix} \frac{1}{\sin\alpha}(R_{\frac{\pi}{2}-m\alpha} + R_{\frac{\pi}{2}-(2n-2-m)\alpha}\sigma_z) & 0 \\ 0 & \frac{1}{\sin\beta}(R_{\frac{\pi}{2}-m\beta} + R_{\frac{\pi}{2}-(2n-m)\beta}\sigma_z) \end{pmatrix}, \tag{4.86}$$

where

$$\sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.87}$$

The expressions for the sums in (4.77),

$$\sum_{n=1}^{m}\langle C_n(m)^T e_i, x\rangle\langle C_n(m)^T e_j, x\rangle \quad\text{and}\quad \sum_{n=1}^{m}\langle D_n(m)^T e_i, x\rangle\langle D_n(m)^T e_j, x\rangle, \tag{4.88}$$

are given in Appendix 1.

4.3.1 The special case $E = 0$

Now we take $E = 0$ so that $\alpha = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$. In this case we choose $m = 6$. This is the smallest natural number so that when $n_1 + n_2$ is an even integer, $m(n_1\alpha + n_2\beta)$ is an integral multiple of 2π. Note that in this case both $m\alpha$ and $m\beta$ are also integral multiple of 2π. Using (4.50), and Appendix 1 we can calculate the expectations of U_kU_l and V_k and then using these together with (4.62), (4.63) and (4.64) we can obtain the expectations $A_k = \mathbb{E}(B_k)$ and $A_{kl} = \mathbb{E}(B_{kl})$. Let $\psi = 2\theta_1 + 2\theta_2 + \frac{\pi}{3}$. Then

$$\mathbb{E}(U_1^2) = \frac{3-\cos^2\theta_3(1+\cos\psi)}{2\cos^4\theta_1\sin^2\theta_3}, \tag{4.89}$$

$$\begin{aligned}\mathbb{E}(V_1) = \Big(\{&-3\sin\theta_1 - \sqrt{3}\cos\theta_1 + 2\sqrt{3}\cos^2\theta_2\cos\theta_1 + 2\sin\theta_2\cos\theta_2\cos\theta_1 \\ &+2\sin\theta_1\cos^2\theta_2 - 2\sin\theta_1\sqrt{3}\sin\theta_2\cos\theta_2\}\cos^2\theta_3 + 6\sin\theta_1\Big)\Big/(4\cos^3\theta_1\sin^2\theta_3).\end{aligned} \tag{4.90}$$

Next we have

$$\mathbb{E}(U_2^2) = \frac{3-\sin^2\theta_3(1+\cos\psi)}{2\cos^4\theta_2\cos^2\theta_3}, \tag{4.91}$$

and

$$\begin{aligned}\mathbb{E}(V_2) = \Big(\{&-3\sin\theta_2 - \sqrt{3}\cos\theta_2 + 2\sqrt{3}\cos^2\theta_1\cos\theta_2 + 2\sin\theta_1\cos\theta_1\cos\theta_2 \\ &+2\sin\theta_2\cos^2\theta_1 - 2\sin\theta_2\sqrt{3}\sin\theta_1\cos\theta_1\}\sin^2\theta_3 + 6\sin\theta_2\Big)\Big/(4\cos^3\theta_2\cos^2\theta_3).\end{aligned} \tag{4.92}$$

Thirdly,

$$\mathbb{E}(V_3) = \frac{3-\cos\psi-4\cos\psi\cos2\theta_3+\cos^2 2\theta_3+3\cos\psi\cos^2 2\theta_3}{8\sin\theta_3\cos^3\theta_3}, \tag{4.93}$$

and

$$\mathbb{E}(U_3^2) = \frac{3 + \cos^2 2\theta_3 - \cos\psi + 3\cos\psi\cos^2 2\theta_3}{4\cos^4\theta_3}. \tag{4.94}$$

We also need the expectations $\mathbb{E}(U_1U_2)$, $\mathbb{E}(U_2U_3)$ and $\mathbb{E}(U_3U_1)$. They are

$$\mathbb{E}(U_1U_2) = \frac{2 - \cos\psi}{\cos^2\theta_1\cos^2\theta_2}, \tag{4.95}$$

$$\mathbb{E}(U_2U_3) = -\frac{\sin\theta_3\sin\psi(-1 + 3\cos^2\theta_3)}{2\cos^3\theta_3\cos^2\theta_2} \tag{4.96}$$

and

$$\mathbb{E}(U_3U_1) = -\frac{\sin\psi(-2 + 3\cos^2\theta_3)}{2\cos^2\theta_1\cos\theta_3\sin\theta_3}\ . \tag{4.97}$$

$$\begin{aligned}
A_1 &= \frac{1}{2} i\cot^2\theta_3\sin\psi, && (4.98)\\
A_2 &= \frac{1}{2} i\tan^2\theta_3\sin\psi, && (4.99)\\
A_3 &= -\frac{1}{4} i(\cot 2\theta_3(1 + 3\cos\psi)\sin^2 2\theta_3 + 2\cot 2\theta_3(\cos\psi - 2)), && (4.100)\\
A_{11} &= \frac{1}{4}(\cot^2\theta_3(\cos\psi - 2) - 3), && (4.101)\\
A_{22} &= \frac{1}{4}(\tan^2\theta_3(\cos\psi - 2) - 3), && (4.102)\\
A_{12} &= \cos\psi - 2, && (4.103)\\
A_{31} &= \sin\psi\frac{-1 + 2\cos 2\theta_3 + 3\cos^2 2\theta_3}{4\sin 2\theta_3}, && (4.104)\\
A_{23} &= \sin\psi\frac{1 + 2\cos 2\theta_3 - 3\cos^2 2\theta_3}{4\sin 2\theta_3}, && (4.105)\\
A_{33} &= \frac{1}{8}((1 + 3\cos\psi)\sin^2 2\theta_3 - 4 - 2\cos\psi). && (4.106)
\end{aligned}$$

The behaviour of these quantities near $\theta_3 = 0$ and near $\theta_3 = \frac{\pi}{2}$ is given in the table below with the notation $\delta = \frac{\pi}{2} - \theta_3$.

	$\theta_3 = 0$	$\theta_3 = \frac{\pi}{2}$
A_1	$\frac{i\sin\psi}{2\theta_3^2}$	0
A_2	0	$\frac{i\sin\psi}{2\delta^2}$
A_3	$-i\frac{\cos\psi - 2}{4\theta_3}$	$i\frac{\cos\psi - 2}{4\delta}$
A_{11}	$\frac{\cos\psi - 2}{4\theta_3^2} - \frac{3}{4}$	$-\frac{3}{4}$
A_{22}	$-\frac{3}{4}$	$\frac{\cos\psi - 2}{4\delta^2} - \frac{3}{4}$
A_{12}	$\cos\psi - 2$	$\cos\psi - 2$
A_{31}	$\frac{\sin\psi}{2\theta_3}$	0
A_{23}	0	$-\frac{\sin\psi}{2\delta}$
A_{33}	$-\frac{\cos\psi + 2}{4}$	$-\frac{\cos\psi + 2}{4}$

By writing $\cos 2n_3\theta_3 \sin^{2s} 2\theta_3$ as a linear combination of terms of the form $\exp(ir\theta_3)$, we can deduce from (4.65) and (4.66) that :

$$\begin{aligned}
&\lim_{\lambda\to 0} \lambda^{-2}\mathbb{E}\left[\exp(i(n_1\theta_1' + n_2\theta_2'))\cos 2n_3\theta_3' \sin^{2s} 2\theta_3' - \exp(i(n_1\theta_1 + n_2\theta_2))\cos 2n_3\theta_3 \sin^{2s} 2\theta_3\right] \\
&= \{-iA_1\frac{\partial}{\partial\theta_1} - iA_2\frac{\partial}{\partial\theta_2} - iA_3\frac{\partial}{\partial\theta_3} - A_{11}\frac{\partial^2}{\partial\theta_1^2} - A_{22}\frac{\partial^2}{\partial\theta_2^2} - A_{33}\frac{\partial^2}{\partial\theta_3^2} - A_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_2} \\
&\quad -A_{23}\frac{\partial^2}{\partial\theta_2\partial\theta_3} - A_{31}\frac{\partial^2}{\partial\theta_3\partial\theta_1}\}\exp(i(n_1\theta_1 + n_2\theta_2))\cos 2n_3\theta_3 \sin^{2s} 2\theta_3. \qquad (4.107)
\end{aligned}$$

Note that if $s \geq 3/2$, the right-hand side of the last equation is continuous in the topology of Ω. Its restriction to $\Omega_0 \cup \Omega_{\frac{\pi}{2}}$ is zero. Moreover, we shall see in Appendix 2 that if $s_1 \geq 3$ and $s_2 \geq 3$ then the first three derivatives of $e^{i(N\theta_1' + M\theta_2')} \sin^{2s_1} \theta_3' \cos^{2s_2} \theta_3'$ with respect to λ are bounded and (1.24) holds. Now $\cos 2n_3\theta_3$ is a polynomial in $\cos 2\theta_3$ and therefore a polynomial in $\sin^2 \theta_3$. Thus we have for $s \geq 3$

$$\begin{aligned}
&\int_{\Omega_{(0,\frac{\pi}{2})}} \{-iA_1\frac{\partial}{\partial\theta_1} - iA_2\frac{\partial}{\partial\theta_2} - iA_3\frac{\partial}{\partial\theta_3} - A_{11}\frac{\partial^2}{\partial\theta_1^2} - A_{22}\frac{\partial^2}{\partial\theta_2^2} - A_{33}\frac{\partial^2}{\partial\theta_3^2} - A_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_2} \\
&-A_{23}\frac{\partial^2}{\partial\theta_2\partial\theta_3} - A_{31}\frac{\partial^2}{\partial\theta_3\partial\theta_1}\}\exp(i(n_1\theta_1 + n_2\theta_2))\cos 2n_3\theta_3 \sin^{2s} 2\theta_3\sigma_0^0(d\theta_1, d\theta_2, d\theta_3) \\
&\qquad = 0. \qquad (4.108)
\end{aligned}$$

If we assume that σ_0^0 is absolutely continuous on $\Omega_{(0,\frac{\pi}{2})}$ with density ρ we get:

$$\int_{\Omega_{(0,\frac{\pi}{2})}} \rho(\theta_1,\theta_2,\theta_3)\{-iA_1\frac{\partial}{\partial\theta_1} - iA_2\frac{\partial}{\partial\theta_2} - iA_3\frac{\partial}{\partial\theta_3} - A_{11}\frac{\partial^2}{\partial\theta_1^2} - A_{22}\frac{\partial^2}{\partial\theta_2^2} - A_{33}\frac{\partial^2}{\partial\theta_3^2}$$
$$-A_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_2} - A_{23}\frac{\partial^2}{\partial\theta_2\partial\theta_3} - A_{31}\frac{\partial^2}{\partial\theta_3\partial\theta_1}\}\exp(i(n_1\theta_1+n_2\theta_2))\cos 2n_3\theta_3 \sin^{2s} 2\theta_3 d\theta_1 d\theta_2 d\theta_3$$
$$= 0. \tag{4.109}$$

Integrating by parts, assuming that $\lim_{\theta_1\to 2\pi}\rho(\theta_1,\theta_2,\theta_3) = \rho(0,\theta_2,\theta_3)$ and $\lim_{\theta_2\to\pi}\rho(\theta_1,\theta_2,\theta_3) = \rho((\theta_1+\pi) \bmod 2\pi, 0, \theta_3)$, we get

$$\int_{\Omega_{(0,\frac{\pi}{2})}} \exp(i(n_1\theta_1+n_2\theta_2))\cos 2n_3\theta_3 \sin^{2s} 2\theta_3\{i\frac{\partial}{\partial\theta_1}A_1 + i\frac{\partial}{\partial\theta_2}A_2 + i\frac{\partial}{\partial\theta_3}A_3 - A_{11}\frac{\partial^2}{\partial\theta_1^2}A_3$$
$$-\frac{\partial^2}{\partial\theta_2^2}A_{22} - \frac{\partial^2}{\partial\theta_3^2}A_{33} - \frac{\partial^2}{\partial\theta_1\partial\theta_2}A_{12} - \frac{\partial^2}{\partial\theta_2\partial\theta_3}A_{23} - \frac{\partial^2}{\partial\theta_3\partial\theta_1}A_{31}\}\rho(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3$$
$$= 0. \tag{4.110}$$

Therefore, since the set of functions $\{e^{i(n_1\theta_1+n_2\theta_2)}\cos 2n_3\theta_3 \sin^{2s} 2\theta_3 \mid n_1, n_2 \in \mathbb{Z}, n_3 \in \mathbb{N}, n_1 + n_2 \text{ even}\}$ is total in the subspace of $C(\Omega)$ consisting of those functions which are zero outside $\Omega_{(0,\frac{\pi}{2})}$,

$$\{i\frac{\partial}{\partial\theta_1}A_1 + i\frac{\partial}{\partial\theta_2}A_2 + i\frac{\partial}{\partial\theta_3}A_3 - \frac{\partial^2}{\partial\theta_1^2}A_{11} - \frac{\partial^2}{\partial\theta_2^2}A_{22}$$
$$-\frac{\partial^2}{\partial\theta_3^2}A_{33} - \frac{\partial^2}{\partial\theta_1\partial\theta_2}A_{12} - \frac{\partial^2}{\partial\theta_2\partial\theta_3}A_{23} - \frac{\partial^2}{\partial\theta_3\partial\theta_1}A_{31}\}\rho(\theta_1,\theta_2,\theta_3) = 0. \tag{4.111}$$

Next we consider the measure on $\Omega_{\frac{\pi}{2}}$ and Ω_0. In the same manner as above we have with $g(\theta_3) = \sin^2\theta_3\cos^4\theta_3$,

$$\lim_{\lambda\to 0}\lambda^{-2}\mathbb{E}\left[g(\theta_3') - g(\theta_3)\right] = \{-iA_3\frac{\partial}{\partial\theta_3} - A_{33}\frac{\partial^2}{\partial\theta_3^2}\}g(\theta_3). \tag{4.112}$$

Here $g(\theta_3)$ has been chosen so that the right-hand side of (4.112) is continuous in the topology of Ω, its restriction to $\Omega_{\frac{\pi}{2}}$ is zero and its restriction to Ω_0 is 2. So, using the results of Appendix 2 again, we have

$$2\sigma_0^0(\Omega_0) + \int_{\Omega_{(0,\frac{\pi}{2})}} \rho(\theta_1,\theta_2,\theta_3)\{-iA_3\frac{\partial}{\partial\theta_3} - A_{33}\frac{\partial^2}{\partial\theta_3^2}\}g(\theta_3)d\theta_1 d\theta_2 d\theta_3 = 0. \tag{4.113}$$

and hence

$$\begin{aligned}
2\sigma_0^0(\Omega_0) &= -\int_{\Omega_{(0,\frac{\pi}{2})}} \rho(\theta_1,\theta_2,\theta_3)\{-iA_3\frac{\partial}{\partial\theta_3} - A_{33}\frac{\partial^2}{\partial\theta_3^2}\}g(\theta_3)d\theta_1 d\theta_2 d\theta_3 \\
&= -\int_{\Omega_{(0,\frac{\pi}{2})}} \rho(\theta_1,\theta_2,\theta_3)\{-iA_1\frac{\partial}{\partial\theta_1} - iA_2\frac{\partial}{\partial\theta_2} - iA_3\frac{\partial}{\partial\theta_3} - A_{11}\frac{\partial^2}{\partial\theta_1^2} - A_{22}\frac{\partial^2}{\partial\theta_2^2} \\
&\quad -A_{33}\frac{\partial^2}{\partial\theta_3^2} - A_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_2} - A_{23}\frac{\partial^2}{\partial\theta_2\partial\theta_3} - A_{31}\frac{\partial^2}{\partial\theta_3\partial\theta_1}\}g(\theta_3)d\theta_1 d\theta_2 d\theta_3.
\end{aligned} \tag{4.114}$$

Integrating (4.114) by parts as above we get

$$\begin{aligned}
2\sigma_0^0(\Omega_0) &= -\int_{\Omega_{(0,\frac{\pi}{2})}} g(\theta_3)\{i\frac{\partial}{\partial\theta_1}A_1 + i\frac{\partial}{\partial\theta_2}A_2 + i\frac{\partial}{\partial\theta_3}A_3 - A_{11}\frac{\partial^2}{\partial\theta_1^2}A_3 - \frac{\partial^2}{\partial\theta_2^2}A_{22} \\
&\quad -\frac{\partial^2}{\partial\theta_3^2}A_{33} - \frac{\partial^2}{\partial\theta_1\partial\theta_2}A_{12} - \frac{\partial^2}{\partial\theta_2\partial\theta_3}A_{23} - \frac{\partial^2}{\partial\theta_3\partial\theta_1}A_{31}\}\rho(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3 \\
&= 0. \qquad (4.115)
\end{aligned}$$

by (4.111). We therefore have

$$\sigma_0^0(\Omega_0) = 0. \tag{4.116}$$

The same argument holds for $\Omega_{\frac{\pi}{2}}$.

Now we return to the equation for ρ. If $\rho = \rho(\psi,\theta_3)$, (4.111) becomes

$$\{2i\frac{\partial}{\partial\psi}(A_1+A_2)+i\frac{\partial}{\partial\theta_3}A_3-4\frac{\partial^2}{\partial\psi^2}(A_{11}+A_{22}+A_{12})-\frac{\partial^2}{\partial\theta_3^2}A_{33}-2\frac{\partial^2}{\partial\theta_3\partial\psi}(A_{23}+A_{31})\}\rho(\psi,\theta_3) = 0, \tag{4.117}$$

where

$$A_1 + A_2 = i\sin\psi(2\cot^2 2\theta_3 + 1), \tag{4.118}$$

$$A_{23} + A_{31} = \sin\psi\cot 2\theta_3, \tag{4.119}$$

$$A_{11} + A_{22} + A_{12} = -\frac{9 - 3\cos\psi - 5\cos^2 2\theta_3 + \cos^2 2\theta_3\cos\psi}{2\sin^2 2\theta_3}, \tag{4.120}$$

$$A_{33} = \frac{1}{8}((1 + 3\cos\psi)\sin^2 2\theta_3 - 4 - 2\cos\psi), \tag{4.121}$$

$$A_3 = -i\frac{1}{8}(\cot 2\theta_3(1 + 3\cos\psi)\sin^2 2\theta_3 + 2\cot 2\theta_3(\cos\psi - 2)). \tag{4.122}$$

If $2\theta_3 = \phi$ and $\rho = \rho(\psi,\phi)$

$$\{i\frac{\partial}{\partial\psi}(A_1+A_2)+i\frac{\partial}{\partial\phi}A_3-2\frac{\partial^2}{\partial\psi^2}(A_{11}+A_{22}+A_{12})-2\frac{\partial^2}{\partial\phi^2}A_{33}-2\frac{\partial^2}{\partial\phi\partial\psi}(A_{23}+A_{31})\}\rho(\psi,\phi) = 0, \tag{4.123}$$

where

$$A_1 + A_2 = i\sin\psi(2\cot^2\phi + 1), \tag{4.124}$$

$$A_{23} + A_{31} = \sin\psi\cot\phi, \tag{4.125}$$

$$A_{11} + A_{22} + A_{12} = -\frac{9 - 3\cos\psi - 5\cos^2\phi + \cos^2\phi\cos\psi}{2\sin^2\phi}, \tag{4.126}$$

$$A_{33} = \frac{1}{8}((1 + 3\cos\psi)\sin^2\phi - 4 - 2\cos\psi), \tag{4.127}$$

$$A_3 = -i\frac{1}{8}(\cot\phi(1 + 3\cos\psi)\sin^2\phi + 2\cot\phi(\cos\psi - 2)). \tag{4.128}$$

With $\rho(\psi,\phi) = \sin\phi\ S(\psi,\phi)$ we can write this differential equation as

$$\begin{aligned}
&-16\sin\phi\cos\phi\sin\psi\frac{\partial^2}{\partial\phi\partial\psi}S(\psi,\phi) \\
&+2\sin^2\phi(\cos^2\phi + 3\cos^2\phi\cos\psi - \cos\psi + 3)\frac{\partial^2}{\partial\phi^2}S(\psi,\phi)
\end{aligned}$$

$$
\begin{aligned}
&+(8\cos^2\phi\cos\psi-24\cos\psi+72-40\cos^2\phi)\frac{\partial^2}{\partial\psi^2}\mathrm{S}(\psi,\phi)\\
&-8\sin\psi(-7+5\cos^2\phi)\frac{\partial}{\partial\psi}S(\psi,\phi)\\
&+2\cos\phi\sin\phi(-17\cos\psi+5\cos^2\phi+15\cos^2\phi\cos\psi-1)\frac{\partial}{\partial\phi}S(\psi,\phi)\\
&-4\sin^2\phi(3\cos^2\phi+9\cos^2\phi\cos\psi-1-9\cos\psi)S(\psi,\phi)=0.
\end{aligned}
\tag{4.129}
$$

The last equation can be simplified to

$$
\begin{aligned}
&-8\sin 2\phi\sin\psi\frac{\partial^2}{\partial\phi\partial\psi}S(\psi,\phi)\\
&-\frac{1}{2}(\cos 2\phi-1)(\cos 2\phi+3\cos\psi\cos 2\phi+\cos\psi+7)\frac{\partial^2}{\partial\phi^2}S(\psi,\phi)\\
&+(4\cos\psi\cos 2\phi-20\cos\psi+52-20\cos 2\phi)\frac{\partial^2}{\partial\psi^2}S(\psi,\phi)\\
&-4\sin\psi(-9+5\cos 2\phi)\frac{\partial}{\partial\psi}S(\psi,\phi)\\
&+\frac{1}{2}\sin 2\phi(15\cos\psi\cos 2\phi+3-19\cos\psi+5\cos 2\phi)\frac{\partial}{\partial\phi}S(\psi,\phi)\\
&+(\cos 2\phi-1)(3\cos 2\phi+9\cos\psi\cos 2\phi+1-9\cos\psi)S(\psi,\phi)=0.
\end{aligned}
\tag{4.130}
$$

This equation can also be written in terms of the variables $u=\cos 2\theta$ and $v=\cos\psi$ as follows:

$$
\begin{aligned}
2(1-u)(1-u^2)(3uv+u+7)\frac{\partial^2}{\partial u^2}S(u,v)&\\
-16(1-u^2)(1-v^2)\frac{\partial^2}{\partial u\partial v}S(u,v)&\\
+4(1-v^2)(uv-5u-5v+13)\frac{\partial^2}{\partial v^2}S(u,v)&\\
-(1-u)(7u^2+21u^2v+22u-2uv-19v+3)\frac{\partial}{\partial u}S(u,v)&\\
+(20uv-52v-36-24uv^2+20u+56v^2)\frac{\partial}{\partial u}S(u,v)&\\
-(1-u)(3u+9uv-9v+1)S(u,v)&=0.
\end{aligned}
\tag{4.131}
$$

Unfortunately, we have not been able to solve this equation, nor prove that it has a unique positive solution.

4.3.2 The case $E\in(-1,1)$ with both α/π and β/π irrational

In this section we consider the case when both α/π and β/π are irrational. We know that in this case σ_0^E is Lebesgue with respect to θ_1 and θ_2, that is, on $\Omega_{(0,\frac{\pi}{2})}$, $\sigma_0^E(d\theta_1,d\theta_2,d\theta_3)=d\theta_1 d\theta_2\tilde{\sigma}_0^E(d\theta_3)$, on $\Omega_{\frac{\pi}{2}}$, $\sigma_0^E(d\theta_1)=\delta_{\frac{\pi}{2}}d\theta_1$ and on Ω_0, $\sigma_0^E(d\theta_2)=\delta_0 d\theta_2$. Here σ_0^E is rotation invariant in both θ_1 and θ_2. Therefore we can choose $m=1$. Also we need only to consider functions of θ_3 to determine the limiting measure.

Since α and β are arbitrary, the expressions for $\mathbb{E}(U_3^2)$ and $\mathbb{E}(V_3)$ are very long. However we only need the integrated expressions with respect to θ_1 and θ_2 . These are much simpler:

$$\int_0^{2\pi}\int_0^{\pi} d\theta_1 d\theta_2 \mathbb{E}(V_3) = \frac{\pi^2}{\sqrt{2}(1+\cos\phi)^{3/2}\sin\theta_3}\left(\frac{1+4\cos\phi-3\cos^2\phi}{\sin^2\beta}+\frac{5-4\cos\phi-\cos^2\phi}{\sin^2\alpha}\right) \tag{4.132}$$

and

$$\int_0^{2\pi}\int_0^{\pi} d\theta_1 d\theta_2 \mathbb{E}(U_3^2) = \frac{2\pi^2}{(1+\cos\phi)^2}\left(\frac{3+4\cos\phi+\cos^2\phi}{\sin^2\beta}+\frac{3-4\cos\phi+\cos^2\phi}{\sin^2\alpha}\right). \tag{4.133}$$

Recall that

$$\mathbb{E}\left(\exp(in_3\theta_3')\right) = \exp(in_3\theta_3)\{1+\lambda^2[A_3 n_3 + A_{33}n_3^2]\} + \mathrm{O}(\lambda^3), \tag{4.134}$$

where $A_3 = \mathbb{E}(B_3)$ and $A_{33} = \mathbb{E}(B_{33})$. Therefore

$$\int_0^{2\pi}\int_0^{\pi} d\theta_1 d\theta_2 \; \mathbb{E}\left(\exp(in_3\theta_3')\right) = \exp(in_3\theta_3)\{1+\lambda^2[C_3 n_3 + C_{33}n_3^2]\} + \mathrm{O}(\lambda^3), \tag{4.135}$$

where $C_3 = \int_0^{2\pi}\int_0^{\pi} A_3 \; d\theta_1 d\theta_2$ and $C_{33} = \int_0^{2\pi}\int_0^{\pi} A_{33} \; d\theta_1 d\theta_2$ and therefore

$$\lim_{\lambda\to 0}\lambda^{-2}\left(\int_0^{2\pi}\int_0^{\pi} d\theta_1 d\theta_2 \; \left(\mathbb{E}\left(\exp(in_3\theta_3')\right) - \exp(in_3\theta_3)\right)\right) = n_3\exp(in_3\theta_3)\left(C_3 + C_{33}n_3\right). \tag{4.136}$$

It follows that for any $g(\theta_3)$ which is a finite linear combination of terms of the form $\exp(in_3\theta_3)$,

$$\lim_{\lambda\to 0}\lambda^{-2}\left(\int_0^{2\pi}\int_0^{\pi} d\theta_1 d\theta_2 \; \left(\mathbb{E}\left(g(\theta_3')\right)\right) - g(\theta_3)\right) = -g''(\theta_3)C_{33} - ig'(\theta_3)C_3. \tag{4.137}$$

If $\lambda^{-2}\|\mathcal{T}_\lambda g - g\|$ is bounded and if the right-hand side of (4.137) is continuous on $[0,\pi/2]$ then we have from (1.24)

$$\int_{(0,\frac{\pi}{2})} \left(g''(\theta_3)C_{33} + ig'(\theta_3)C_3\right)\tilde{\sigma}_0^E(d\theta_3) + \pi\delta_0\left(g''(\theta_3)C_{33} + ig'(\theta_3)C_3\right)\Big|_{\theta_3=0}$$
$$+\pi\delta_{\frac{\pi}{2}}\left(g''(\theta_3)C_{33} + ig'(\theta_3)C_3\right)\Big|_{\theta_3=\pi/2} = 0. \tag{4.138}$$

In terms of $\phi = 2\theta_3$ we have

$$C_3 = i\frac{\pi^2}{8}\left(\frac{(1-\cos\phi)(5\cos\phi-\cos^2\phi+2)}{\sin^2\alpha}+\frac{(1+\cos\phi)(5\cos\phi+\cos^2\phi-2)}{\sin^2\beta}\right)\operatorname{cosec}\phi, \tag{4.139}$$

$$C_{33} = -\frac{\pi^2}{16}\left(\frac{3+4\cos\phi+\cos^2\phi}{\sin^2\beta}+\frac{3-4\cos\phi+\cos^2\phi}{\sin^2\alpha}\right). \tag{4.140}$$

In Appendix 2 we show that if $g(\theta_3) = f(\sin^2\theta_3)$ and the first three derivatives of f are bounded, then (1.24) holds. Now, if $g(\theta_3)$ is a linear combination of terms of the form $\cos 2n_3\theta_3$ then it is a polynomial in $\cos 2\theta_3$ and therefore a polynomial in $\sin^2\theta_3$. Let

$$g(\theta_3) = \frac{1}{4(n_3-2)}\cos 2(n_3-2)\theta_3 + \frac{1}{4(n_3+2)}\cos 2(n_3+2)\theta_3 - \frac{1}{2n_3}\cos 2n_3\theta_3, \tag{4.141}$$

for $n_3 \neq 0$ and $n_3 \neq \pm 2$. For $n_3 = \pm 2$ we take

$$g(\theta_3) = \pm\left(\frac{1}{16}\cos 8\theta_3 - \frac{1}{4}\cos 4\theta_3\right). \tag{4.142}$$

Then

$$g'(\theta_3) = 2\sin 2n_3\theta_3 \sin^2 2\theta_3 \tag{4.143}$$

and (4.138) becomes for $n_3 \neq 0$,

$$\int_{(0,\frac{\pi}{2})} \left[2n_3C_{33}\cos 2n_3\theta_3 \sin^2 2\theta_3 + \left(iC_3\sin^2 2\theta_3 + 4C_{33}\sin 2\theta_3\cos 2\theta_3\right)\sin 2n_3\theta_3\right]\tilde{\sigma}_0^E(d\theta_3) = 0. \tag{4.144}$$

We can integrate by parts to get:

$$\begin{aligned}&\int_{(0,\frac{\pi}{2})} \left(iC_3\sin^2 2\theta_3 + 4C_{33}\sin 2\theta_3\cos 2\theta_3\right)\sin 2n_3\theta_3\,\tilde{\sigma}_0^E(d\theta_3)\\ &= -2n_3\int_{(0,\frac{\pi}{2})}\left(\int_{(0,\theta)}\left(iC_3(\theta_3')\sin^2\theta_3' + 4C_{33}(\theta_3')\sin 2\theta_3'\cos 2\theta_3'\right)\tilde{\sigma}_0^E(d\theta_3')\right)\cos 2n_3\theta_3\,d\theta_3.\end{aligned} \tag{4.145}$$

Therefore

$$\begin{aligned}&\int_{(0,\frac{\pi}{2})} C_{33}\cos 2n_3\theta_3\sin^2 2\theta_3\,\tilde{\sigma}_0^E(d\theta_3)\\ &= \int_{(0,\frac{\pi}{2})}\left(\int_{(0,\theta)}\left(iC_3(\theta_3')\sin^2 2\theta_3' + 4C_{33}(\theta_3')\sin 2\theta_3'\cos 2\theta_3'\right)\tilde{\sigma}_0^E(d\theta_3')\right)\cos 2n_3\theta_3\,d\theta_3.\end{aligned} \tag{4.146}$$

Since the set $\{\cos 2n_3\theta_3 \mid n_3 \in \mathbb{N}_0\}$ is total in $C([0,\frac{\pi}{2}])$,

$$C_{33}\sin^2 2\theta_3\,\tilde{\sigma}_0^E(d\theta_3) = \left(\int_{(0,\theta)}\left(iC_3(\theta_3')\sin^2 2\theta_3' + 4C_{33}(\theta_3')\sin 2\theta_3'\cos 2\theta_3'\right)\tilde{\sigma}_0^E(d\theta_3')\right)d\theta_3 + Kd\theta_3, \tag{4.147}$$

where K is a constant. $C_{33}\sin^2 2\theta_3$ never vanishes on $(0,\frac{\pi}{2})$, therefore $\tilde{\sigma}_0^E$ is absolutely continuous and if its density is ρ,

$$C_{33}\sin^2 2\theta_3\rho(\theta_3) = \int_{(0,\theta)}\left(iC_3(\theta_3')\sin^2 2\theta_3' + 4C_{33}(\theta_3')\sin 2\theta_3'\cos 2\theta_3'\right)\rho(\theta_3')d\theta_3' + K. \tag{4.148}$$

It follows that ρ is differentiable and

$$\frac{d}{d\theta_3}(C_{33}\sin^2 2\theta_3\rho(\theta_3)) = \left(iC_3(\theta_3)\sin^2 2\theta_3 + 4C_{33}(\theta_3)\sin 2\theta_3\cos 2\theta_3\right)\rho(\theta_3) \tag{4.149}$$

or

$$\frac{d}{d\theta_3}(C_{33}\rho(\theta_3)) - iC_3\rho(\theta_3) = 0. \tag{4.150}$$

We shall solve this equation below, but first, as in the case $E = 0$, let $g(\theta) = \sin^2\theta_3\cos^4\theta_3$. Then (4.138) becomes, for $n_3 \neq 0$,

$$\int_0^{\frac{\pi}{2}} \left(g''(\theta_3)C_{33} + ig'(\theta_3)C_3\right)\rho(\theta_3)d\theta_3 = \frac{2\pi^3}{\sin^2\beta}\delta_0. \tag{4.151}$$

Since ρ satisfies the differential equation (4.150) the left-hand side of the last equation vanishes and therefore $\delta_0 = 0$. Similarly $\delta_{\frac{\pi}{2}} = 0$.

We now proceed to solve the differential equation (4.150). If $\rho = \rho(\phi)$,

$$iC_3\rho(\phi) - 2\frac{d}{d\phi}\left(C_{33}\rho(\phi)\right) = 0. \tag{4.152}$$

Simplifying and putting $\rho(\phi) = R(\phi)\sin\phi$, we get

$$\left(\frac{(1-\cos\phi)(3-\cos\phi)}{\sin^2\alpha} + \frac{(1+\cos\phi)(\cos\phi+3)}{\sin^2\beta}\right)\frac{dR(\phi)}{d\phi}$$
$$+2\sin\phi\left(\frac{1-\cos\phi}{\sin^2\alpha} - \frac{1+\cos\phi}{\sin^2\beta}\right)R(\phi) = 0. \tag{4.153}$$

With $R(\phi) = S(\cos\phi)$ and $t = \cos\phi$ this becomes

$$\left(\frac{(1-t)(3-t)}{\sin^2\alpha} + \frac{(1+t)\,(t+3)}{\sin^2\beta}\right)\frac{dS(t)}{dt} - 2\left(\frac{1-t}{\sin^2\alpha} - \frac{1+t}{\sin^2\beta}\right)S(t) = 0 \tag{4.154}$$

or

$$\left((1-t)(3-t)\sin^2\beta + (1+t)(3+t)\sin^2\alpha\right)\frac{dS(t)}{dt} - 2\left((1-t)\sin^2\beta - (1+t)\sin^2\alpha\right)S(t) = 0. \tag{4.155}$$

$$\frac{d}{dt}\left[\left((1-t)(3-t)\sin^2\beta + (1+t)(3+t)\sin^2\alpha\right)S(t)\right] - 2\left(\cos^2\beta - \cos^2\alpha\right)S(t) = 0. \tag{4.156}$$

$$\frac{d}{dt}\left[\frac{\left((1-t)(3-t)\sin^2\beta + (1+t)(3+t)\sin^2\alpha\right)}{\sin^2\beta+\sin^2\alpha}S(t)\right] - 2\left(\frac{\cos^2\beta-\cos^2\alpha}{\sin^2\beta+\sin^2\alpha}\right)S(t) = 0. \tag{4.157}$$

Let

$$\frac{\left((1-t)(3-t)\sin^2\beta + (1+t)(3+t)\sin^2\alpha\right)}{\sin^2\beta+\sin^2\alpha} = (t-t_+)(t-t_-) \tag{4.158}$$

and

$$U(t) = (t-t_+)(t-t_-)S(t) \tag{4.159}$$

then

$$\frac{dU(t)}{dt} - 2\left(\frac{\cos^2\beta-\cos^2\alpha}{\sin^2\beta+\sin^2\alpha}\right)\frac{U(t)}{(t-t_+)(t-t_-)} = 0. \tag{4.160}$$

Recall that $\cos^2\beta = \cos^2\alpha$ only if $E = 0$ and therefore t_+ and t_- are not pure imaginary. From now on it is easier to work in terms of E.

$$\frac{dU(t)}{dt} = \left(\frac{4E}{3-E^2}\right)\frac{U(t)}{(t-t_+)(t-t_-)}, \tag{4.161}$$

where t_+ and t_- are the solutions of $t^2 - \frac{8E}{3-E^2}t + 3 = 0$, that is

$$t_\pm = \frac{4E}{3-E^2} \pm \frac{\sqrt{34E^2 - 3E^4 - 27}}{3-E^2}. \tag{4.162}$$

Note that

$$U(-E,t) = U(E,-t) \tag{4.163}$$

and therefore we only need look at the case $E > 0$. Let $E_0 = (\sqrt{13}-2)/\sqrt{3} \approx 0.927$. In the case when $E = E_0$, $t_+ = t_- = a$ where $a = 4E_0/(3-E_0^2)$ and then

$$S(t) = \frac{C}{(a-t)^2}\exp\left(\frac{a}{a-t}\right). \tag{4.164}$$

(Note that $a > 1.7$.)

In the case $E_0 < E < 1$, $t_\pm = a \pm b$ where $a = \dfrac{4E}{(3-E^2}$ and $b = \dfrac{\sqrt{34E^2 - 3E^4 - 27}}{3-E^2}$ and

$$S(t) = \frac{C}{(a-t)^2 - b^2}\left(\frac{a+b-t}{a-b-t}\right)^{\frac{a}{2b}}. \tag{4.165}$$

In the case $0 < E < E_0$, $t_\pm = a \pm ib$ where $a = \dfrac{4E}{3-E^2}$ and $b = \dfrac{\sqrt{3E^4 - 34E^2 + 27}}{3-E^2}$ and

$$S(t) = \frac{C}{(a-t)^2 + b^2}\exp\left(\frac{a}{b}\tan^{-1}\frac{b}{(a-t)}\right). \tag{4.166}$$

Notice the limiting cases

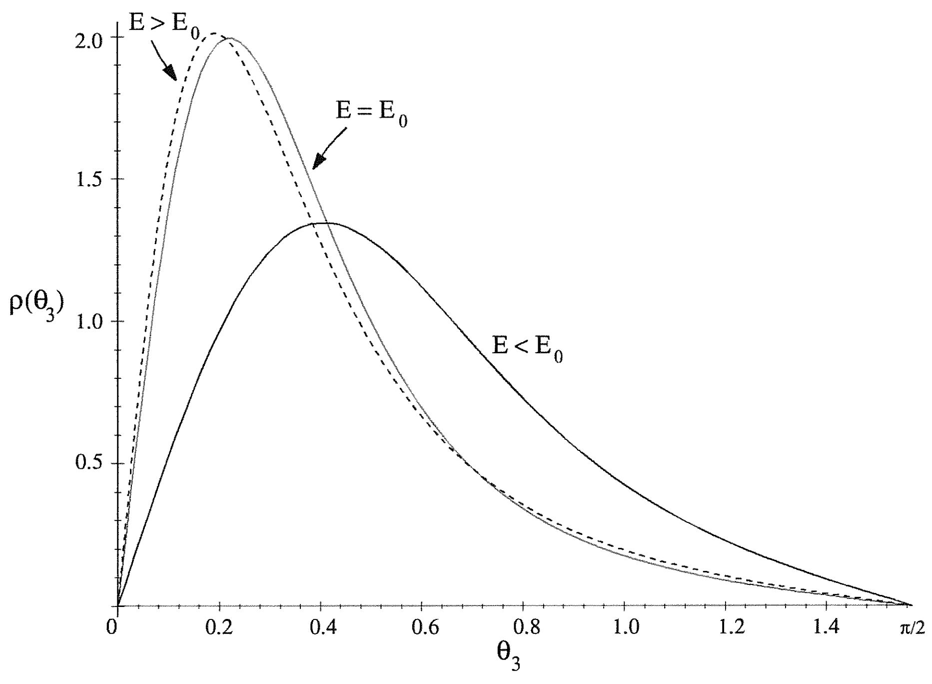

Figure 2: $\theta_3 \mapsto \rho(\theta_3)$

$$E \to 0 \Longrightarrow S(t) \to \frac{C}{t^3+3} \tag{4.167}$$

and

$$E \to 1 \Longrightarrow S(t) \to \frac{C}{(1-t)^2}. \tag{4.168}$$

The former clearly does not satisfy the equation (4.130), which means that there is an anomaly at $E = 0$. The second even diverges at $t = 1$ and the corresponding $\rho(\theta_3)$ also diverges at $\theta_3 = 0$. This of course means that the constant C needs to be scaled and the resulting measure is Lebesgue measure on Ω_0. This is due to the fact that the coordinates are singular at this point, however, and we need a more careful analysis. For small ϵ we can write $a \approx 2(1-2\epsilon)$

and $b \approx 1 - 8\epsilon$, so that $S(t) \approx \frac{C}{1+4\epsilon-t)^2}$, replacing $a/2b$ by 1. The normalisation constant C must be proportional to ϵ, so the density is

$$\rho(\theta_3) \sim \frac{C\epsilon \sin 2\theta_3}{1 + 4\epsilon - \cos 2\theta_3}. \tag{4.169}$$

To compare this measure with the invariant measure at $E = 1$ we need to change coordinates. The corresponding transformation is given by $S_1 S^{-1}$, where S_1 is the matrix (4.200) and S is the matrix (4.69. For $E = 1 - \epsilon$ we have

$$S^{-1} \approx \frac{1}{2}\begin{pmatrix} 1 & \epsilon^{-1/2} & 0 & -1 \\ -1 & -\epsilon^{-1/2} & 0 & -1 \\ 0 & \epsilon^{-1/2} & 1 & 0 \\ 0 & -\epsilon^{-1/2} & 1 & 1 \end{pmatrix}. \tag{4.170}$$

Thus

$$S_1 S^{-1} \approx \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -1/\sqrt{\epsilon} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.171}$$

and hence if we denote the original coordinates by θ and the new coordinates by θ',

$$cot\theta'_1 = -\cot(\theta_2 - \pi/4), \tag{4.172}$$

$$\cot\theta'_2 \approx -\sqrt{\epsilon}\tan\theta_1 \tag{4.173}$$

and

$$\cot^2\theta'_3 \approx \frac{1}{2}(\sin^2\theta_1 + \epsilon^{-1}\cos^2\theta_1)\tan^2\theta_3. \tag{4.174}$$

It follows that $d\theta_2 = d\theta'_1$ and

$$d\theta_1 = \frac{1}{\sqrt{\epsilon}}\frac{\cos^2\theta_1}{\sin^2\theta'_2}d\theta'_2 = \frac{\sqrt{\epsilon}d\theta'_2}{\cos^2\theta'_2 + \epsilon\sin^2\theta'_2}, \tag{4.175}$$

and

$$\frac{d\theta'_3}{\sin^2\theta'_3} = \frac{1}{\sqrt{2}}\left(\sin^2\theta_1 + \epsilon^{-1}\cos^2\theta_1\right)^{1/2}\frac{d\theta'_3}{\cos^2\theta'_3}. \tag{4.176}$$

We also have

$$\sin^2\theta_1 + \epsilon^{-1}\cos^2\theta_1 = \frac{1}{\cos^2\theta'_2 + \epsilon\sin^2\theta'_2}. \tag{4.177}$$

Denoting

$$X = \cos^2\theta'_2 + \epsilon\sin^2\theta'_2 \tag{4.178}$$

we have

$$d\theta_3 = \sqrt{2}\frac{d\theta'_3}{\sin^2\theta'_3}X^{1/2}\cos^2\theta_3 = \sqrt{2}\frac{d\theta'_3}{\sin^2\theta'_3}\frac{X^{1/2}}{1 + 2X\cot^2\theta'_3}. \tag{4.179}$$

Similarly, transforming the density, we have

$$\sin 2\theta_3 = \frac{2\sqrt{2}\cot\theta'_3 X^{1/2}}{1 + 2X\cot^2\theta'_3} \tag{4.180}$$

and

$$1 - \cos 2\theta'_3 = \frac{4X\cot^2\theta'_3}{1 + 2X\cot^2\theta'_3}, \tag{4.181}$$

so that

$$\rho(\theta_3') = \frac{2\sqrt{2}X^{1/2}\cot\theta_3'(1+2X\cot^2\theta_3')}{[4X\cot^2\theta_3' + 4\epsilon(1+2X\cot^2\theta_3')]^2}. \tag{4.182}$$

We thus get

$$\begin{aligned}\rho(\theta_3)d\theta_1 d\theta_2 d\theta_3 &= \frac{C\epsilon\sqrt{\epsilon}\cot\theta_3'\, d\theta_1' d\theta_2' d\theta_3'}{4\left[\cot^2\theta_3'\cos^2\theta_2' + \epsilon(1+\cot^2\theta_3'(1+\cos^2\theta_2'))\right]^2} &(4.183)\\ &= \frac{C\epsilon\sqrt{\epsilon}\sin\theta_3'\cos\theta_3'\, d\theta_1' d\theta_2' d\theta_3'}{4\left[\cos^2\theta_2'\cos^2\theta_3' + \epsilon(\sin^2\theta_3' + \cos^2\theta_3'(1+\cos^2\theta_2'))\right]^2}. &(4.184)\end{aligned}$$

In the limit $\epsilon \to 0$ this tends to

$$\nu_1 = \delta(\theta_2' - \frac{1}{2}\pi)\sin\theta_3'\, d\theta_1' d\theta_3'. \tag{4.185}$$

4.3.3 The case $E \in (-1,1)$ with α/π rational and β/π irrational

In this section we consider the case when α/π is rational and β/π is irrational. We know that in this case σ_0^E is Lebesgue with respect to θ_2, that is, on $\Omega_{(0,\frac{\pi}{2})}$, $\sigma_0^E(d\theta_1, d\theta_2, d\theta_3) = d\theta_2\hat{\sigma}_0^E(d\theta_1, d\theta_3)$, and on Ω_0, $\sigma_0^E(d\theta_2) = \delta_0 d\theta_2$. Since we need to consider only functions of θ_1 and θ_3 to determine the limiting measure we choose m so that $m\alpha$ is an integral multiple of π. The quantities that we need are:

$$\int_0^\pi d\theta_2 \mathbb{E}(U_1^2) = 2m\pi \frac{2\sin^2\alpha\cos^2\theta_3 + 3\sin^2\beta\sin^2\theta_3}{\cos^4\theta_1\sin^2\alpha\sin^2\beta\sin^2\theta_3}, \tag{4.186}$$

$$\int_0^\pi d\theta_2 \mathbb{E}(V_1) = 2\sin\theta_1 m\pi \frac{(\sin^2\alpha\cos^2\theta_3 + 3\sin^2\beta\sin^2\theta_3}{\cos^3\theta_1\sin^2\alpha\sin^2\beta\sin^2\theta_3}, \tag{4.187}$$

$$\int_0^\pi d\theta_2 \mathbb{E}(V_3) = \frac{1}{2}\pi m \frac{(1+\cos\phi)(1+3\cos\phi)\sin^2\alpha + (5+\cos\phi)(1-\cos\phi)\sin^2\beta}{\sin^2\alpha\sin^2\beta\sin\phi\cos^2\theta_3}, \tag{4.188}$$

$$\int_0^\pi d\theta_2 \mathbb{E}(U_3^2) = \frac{1}{2}\pi m \frac{(3+4\cos\phi+\cos^2\phi)\sin^2\alpha + (3-4\cos\phi+\cos^2\phi)\sin^2\beta}{\sin^2\alpha\sin^2\beta\cos^4\theta_3}, \tag{4.189}$$

$$\int_0^\pi d\theta_2 \mathbb{E}(U_3 U_1) = 0. \tag{4.190}$$

Starting from

$$\begin{aligned}\mathbb{E}\left(\exp(i(n_1\theta_1' + n_3\theta_3'))\right) &= \exp(i(n_1\theta_1 + n_3\theta_3))\exp(-imn_1\alpha)\\ &\quad \times\{1 + \lambda^2[A_1 n_1 + A_3 n_3 + A_{11}n_1^2 + A_{33}n_3^2 + A_{31}n_3 n_1]\}\\ &\quad + O(\lambda^3),\end{aligned}$$

where $A_k = \mathbb{E}(B_k)$ and $A_{kl} = \mathbb{E}(B_{kl})$, we get

$$\begin{aligned}\int_0^\pi d\theta_2 \mathbb{E}\left(\exp(i(n_1\theta_1' + n_3\theta_3'))\right) &= \exp(i(n_1\theta_1 + n_3\theta_3))\exp(-imn_1\alpha)\\ &\quad \times\{1 + \lambda^2[C_1 n_1 + C_3 n_3 + C_{11}n_1^2 + C_{33}n_3^2 + C_{31}n_3 n_1]\}\\ &\quad + O(\lambda^3),\end{aligned}$$

where $C_k = \int_0^\pi d\theta_2 A_k$ and $C_{kl} = \int_0^\pi d\theta_2 A_{kl}$. Therefore

$$\lim_{\lambda\to 0} \lambda^{-2}\left(\int_0^\pi d\theta_2\ \mathbb{E}\left(\exp(i(n_1\theta_1' + n_3\theta_3')) - \exp(i(n_1\theta_1 + n_3\theta_3))\exp(-imn_1\alpha)\right)\right)$$
$$= \exp(i(n_1\theta_1 + n_3\theta_3))\exp(-imn_1\alpha)[C_1n_1 + C_3n_3 + C_{11}n_1^2 + C_{33}n_3^2 + C_{31}n_3n_1]$$

and

$$\lim_{\lambda\to 0} \lambda^{-2}\left(\int_0^\pi d\theta_2\ \mathbb{E}\left(\exp(2i(n_1\theta_1' + n_3\theta_3')) - \exp(2i(n_1\theta_1 + n_3\theta_3))\right)\right)$$
$$= 2\exp(2i(n_1\theta_1 + n_3\theta_3))[C_1n_1 + C_3n_3 + 2C_{11}n_1^2 + 2C_{33}n_3^2 + 2C_{31}n_3n_1]. \tag{4.191}$$

It turns out that C_1 and C_{31} are both zero and C_3 and C_{33} are πm times their values in the previous case. There remains C_{11} which is given below. Note that it is independent of θ_1.

$$C_{11} = -2m\pi\frac{2(1+\cos\phi)\sin^2\alpha + 3(1-\cos\phi)\sin^2\beta}{(1-\cos\phi)\sin^2\alpha\sin^2\beta}. \tag{4.192}$$

For suitable functions g such that

$$\frac{\partial^2 g}{\partial\theta_3^2}(\theta_1,\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_1,\theta_3)C_3 + \frac{\partial^2 g}{\partial\theta_1^2}(\theta_1,\theta_3)C_{11} \tag{4.193}$$

is continuous we have as in previous cases.

$$\int_{[0,2\pi)\times(0,\frac{\pi}{2})}\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_1,\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_1,\theta_3)C_3 + \frac{\partial^2 g}{\partial\theta_1^2}(\theta_1,\theta_3)C_{11}\right)\hat{\sigma}_0^E(d\theta_1,d\theta_3)$$
$$+\ \pi\delta_0\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_1,\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_1,\theta_3)C_3 + \frac{\partial^2 g}{\partial\theta_1^2}(\theta_1,\theta_3)C_{11}\right)\Big|_{\theta_3=0}$$
$$+\int_{[0,\pi)}\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_1,\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_1,\theta_3)C_3 + \frac{\partial^2 g}{\partial\theta_1^2}(\theta_1,\theta_3)C_{11}\right)\Big|_{\theta_3=\pi/2}\sigma_0^E(d\theta_1) = 0. \tag{4.194}$$

Note that

$$\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_1,\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_1,\theta_3)C_3 + \frac{\partial^2 g}{\partial\theta_1^2}(\theta_1,\theta_3)C_{11}\right)\Big|_{\theta_3=0} \tag{4.195}$$

is independent of θ_1. If we assume that σ_0^E is absolutely continuous on $\Omega_{(0,\frac{\pi}{2})}$ with density ρ by choosing g's whose restriction to $\Omega_0 \cup \Omega_{\frac{\pi}{2}}$ is zero and integrating by parts, we get with $\hat{\rho}(\theta_1,\theta_3) = \int_{[0,\pi)}\rho(\theta_1,\theta_2,\theta_3)d\theta_2$:

$$\left(\frac{\partial^2}{\partial\theta_3^2}C_{33} - i\frac{\partial}{\partial\theta_3}C_3 + \frac{\partial^2}{\partial\theta_1^2}C_{11}\right)\hat{\rho}(\theta_1,\theta_3) = 0. \tag{4.196}$$

As in the previous two cases we can show that σ_0^E vanishes on $\Omega_0 \cup \Omega_{\frac{\pi}{2}}$. If g is independent of θ_1 and $\bar{\sigma}_0^E(d\theta_3) = \int_{[0,2\pi)}\hat{\sigma}_0^E(d\theta_1,d\theta_3)$, then (4.194) becomes

$$\int_{(0,\frac{\pi}{2})}\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_3)C_3\right)\bar{\sigma}_0^E(d\theta_3)$$
$$+\ \pi\delta_0\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_3)C_3\right)\Big|_{\theta_3=0}$$
$$+\pi\delta_{\frac{\pi}{2}}\left(\frac{\partial^2 g}{\partial\theta_3^2}(\theta_3)C_{33} + i\frac{\partial g}{\partial\theta_3}(\theta_3)C_3\right)\Big|_{\theta_3=\pi/2} = 0 \tag{4.197}$$

and therefore $\bar{\sigma}_0^E$ coincides with σ_0^E in the previous case.

4.4 The case $E = \pm 1$

Suppose that $E = 1$; the case $E = -1$ is similar. Here the real Jordan form for A_0 is

$$J_0 = SA_0S^{-1} = \begin{pmatrix} R_{\frac{\pi}{2}} & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix} \tag{4.198}$$

where

$$\mathcal{J}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.199}$$

The matrix S is then given by

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{4.200}$$

Note that

$$\mathcal{J}_2^q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \tag{4.201}$$

and therefore $\theta_1^{(q)}$, $\theta_2^{(q)}$ and $\theta_3^{(q)}$ are given by

$$\theta_1^{(q)} = (\theta_1 - \frac{q\pi}{2}) \bmod 2\pi, \tag{4.202}$$

$$\cot \theta_2^{(q)} = \begin{cases} \frac{1}{q} & \text{if } \theta_2 = 0, \\ \frac{\cot\theta_2}{1+q\cot\theta_2}, & \text{if } \theta_2 \neq 0, \end{cases} \tag{4.203}$$

and

$$\cot\theta_3^{(q)} = \cot\theta_3(1 + q\sin\theta_2\cos\theta_2 + q^2\cos\theta_2)^{\frac{1}{2}}. \tag{4.204}$$

Therefore $\theta_2^{(q)} \to \frac{\pi}{2}$ as $q \to \infty$. If $\theta_3^{(q)} = 0$ or $\frac{\pi}{2}$, then $\theta_3^{(q)} = \theta_3$. If $\theta_2 = \frac{\pi}{2}$, then $\theta_3^{(q)} = \theta_3$, otherwise $\theta_3^{(q)} \to 0$.

We have

$$\int_\Omega g(\omega)\sigma_0^E(d\omega) = \lim_{q\to\infty}\int_\Omega (\mathcal{T}_0^{4q}g)(\omega)\sigma_0^1(d\omega). \tag{4.205}$$

By using the functions (4.33) in (4.205), we get for $n_1, n_2 \in \mathbb{Z}$, $n_3 \in \mathbb{N}$, $n_1 + n_2$ even,

$$\int_{\Omega_{(0,\frac{\pi}{2})}} e^{i(n_1\theta_1+n_2\theta_2)}\sin 2n_3\theta_3\,\sigma_0^E(d\theta_1 d\theta_2 d\theta_3) = \int_{\Omega_{(0,\frac{\pi}{2})}\cap\{\theta_2=\frac{\pi}{2}\}} e^{i(n_1\theta_1+n_2\theta_2)}\sin 2n_3\theta_3\,\sigma_0^1(d\theta_1 d\theta_2 d\theta_3). \tag{4.206}$$

Thus σ_0^1 on $\Omega_{(0,\frac{\pi}{2})}$ is concentrated on $\Omega_{(0,\frac{\pi}{2})} \cap \{\theta_2 = \frac{\pi}{2}\}$. Then by using the functions (4.35) in (4.205), we get, for $n_2 \in \mathbb{Z}$,

$$\int_{\Omega_0} e^{2in_2\theta_2}\sigma_0^1(d\theta_2) = \int_{\Omega_0\cap\{\theta_2=\frac{\pi}{2}\}} e^{2in_2\theta_2}\sigma_0^1(d\theta_2). \tag{4.207}$$

Therefore σ_0^1 is concentrated on $(\Omega_{(0,\frac{\pi}{2})} \cup \Omega_0) \cap \{\theta_2 = \frac{\pi}{2}\} \cup \Omega_{\frac{\pi}{2}}$.

Since

$$(\mathcal{T}_0^4 g)(\theta_1, \frac{\pi}{2}, \theta_3) = g((\theta_1, \frac{\pi}{2}, \theta_3) \tag{4.208}$$

we have

$$\int_\Omega \left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^4 g\right)_{\lambda=0}(\theta)\,\sigma_0^1(d\theta) = 0. \tag{4.209}$$

It is sufficient to calculate $\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^4 g\right)_{\lambda=0}(\theta_1, \frac{\pi}{2}, \theta_3)$. Let

$$x = \begin{pmatrix} \sin\theta_1 \sin\theta_3 \\ \cos\theta_1 \sin\theta_3 \\ \cos\theta_3 \\ 0 \end{pmatrix} \tag{4.210}$$

and let

$$x' = \tilde{B}(4)\,x = \begin{pmatrix} \sin\theta_1' \sin\theta_3' \\ \cos\theta_1' \sin\theta_3' \\ \sin\theta_2' \cos\theta_3' \\ \cos\theta_2' \cos\theta_3' \end{pmatrix}. \tag{4.211}$$

Then we get from (4.65) with $n_2 = 0$ and $m = 4$

$$\begin{aligned}\mathbb{E}\left(\exp(i(n_1\theta_1' + n_3\theta_3'))\right) &= \exp(i(n_1\theta_1 + n_3\theta_3)) \\ &\times\left\{1 + \lambda^2[A_1n_1 + A_3n_3 + A_{11}n_1^2 + A_{33}n_3^2 + A_{31}n_3n_1]\right\} + \mathrm{O}(\lambda^3).\end{aligned} \tag{4.212}$$

To calculate the A's we need:

$$C_1(4) = \begin{pmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}, \quad C_2(4) = \begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}, \tag{4.213}$$

$$C_3(4) = \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad C_4(4) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ \frac{3}{2} & \frac{3}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \tag{4.214}$$

and

$$D_1(4) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \quad D_2(4) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \tag{4.215}$$

$$D_3(4) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad D_4(4) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{4.216}$$

Using (4.50) we get from these, with $\psi = 2\theta_1$ and $\phi = 2\theta_3$,

$$\begin{aligned}
\mathbb{E}(y_1^2) &= \mathbb{E}(y_2^2) = \mathbb{E}(y_4^2) = \frac{5 + 3\cos\phi}{4} \\
\mathbb{E}(y_1y_2) &= -\frac{1}{4}\sin\psi(1 - \cos\phi) \\
\mathbb{E}(y_1y_3) &= -\frac{1}{2}(\sin\theta_1 + 3\cos\theta_1)\sin\phi \\
\mathbb{E}(y_2y_3) &= \frac{1}{2}(3\sin\theta_1 + \cos\theta_1)\sin\phi \\
\mathbb{E}(y_3^2) &= \frac{1}{8}(35 + 21\cos\phi + 3(1 - \cos\phi)\sin\psi).
\end{aligned} \tag{4.217}$$

These give

$$A_1 = \frac{i}{2}\cos\psi\sin\psi, \tag{4.218}$$

$$\begin{aligned} A_3 &= \frac{i}{16}\Big\{\frac{4(5\cos\phi+3)}{\sin\phi} + 13\sin\phi + 15\cos\phi\sin\phi + 2(2-\cos\phi)\sin\phi\sin^2\psi \\ &\qquad -8\cos\phi\sin\phi\cos\psi + 3(1-\cos\phi)\sin\phi\sin\psi\Big\}, \end{aligned} \tag{4.219}$$

$$A_{11} = \frac{(3-\sin^2\psi)}{4} - \frac{2}{1-\cos\phi}, \tag{4.220}$$

$$A_{31} = \frac{\sin\phi}{4}(\cos\psi\sin\psi - 6 - 2\sin\psi), \tag{4.221}$$

and

$$\begin{aligned} A_{33} &= -\frac{1}{32}\Big((2\sin^2\psi + 3\sin\psi + 8\cos\psi - 15)\cos^2\phi + (2-6\sin\psi)\cos\phi \\ &\quad +(3\sin\psi - 8\cos\psi - 2\sin^2\psi + 45)\Big). \end{aligned} \tag{4.222}$$

As in the previous cases we then get for suitable g's

$$\int_{\Omega\cap\{\theta_2=\frac{\pi}{2}\}} \{-iA_1\frac{\partial}{\partial\theta_1} - iA_3\frac{\partial}{\partial\theta_3} - A_{11}\frac{\partial^2}{\partial\theta_1^2} - A_{33}\frac{\partial^2}{\partial\theta_3^2} - A_{31}\frac{\partial^2}{\partial\theta_3\partial\theta_1}\}g(\theta_1,\theta_3)\sigma_0^1(d\theta_1 d\theta_3) = 0. \tag{4.223}$$

If we assume that σ_0^1 restricted to $\Omega_{(0,\frac{\pi}{2})}\cap\{\theta_2=\frac{\pi}{2}\}$ is absolutely continuous with density ρ, then choosing g's whose restriction to $\Omega_0\cup\Omega_{\frac{\pi}{2}}$ is zero and such that the integrand is continuous, by integrating by parts we can show that ρ satisfies the differential equation

$$\{i\frac{\partial}{\partial\theta_1}A_1 + i\frac{\partial}{\partial\theta_3}A_3 - \frac{\partial^2}{\partial\theta_1^2}A_{11} - \frac{\partial^2}{\partial\theta_3^2}A_{33} - \frac{\partial^2}{\partial\theta_3\partial\theta_1}A_{31}\}\rho(\theta_1,\theta_3) = 0. \tag{4.224}$$

Near $\theta_3 = 0$, A_3 behaves like $i\theta_3^{-1} + O(\theta_3)$ and A_{33} behaves like $-1 + O(\theta_3^2)$. While near $\theta_3 = \frac{\pi}{2}$, $A_3 = -i(4(\frac{\pi}{2}-\theta_3))^{-1} + O((\frac{\pi}{2}-\theta_3))$ and

$$A_{33} = -\frac{1}{8}(3\sin\psi + 7) + O((\frac{\pi}{2}-\theta_3)^2). \tag{4.225}$$

Therefore, by choosing $g(\theta_3) = \sin^2\theta_3\cos^4\theta_3$ we see that the measure σ_0^1 is zero on Ω_0 and by choosing $g(\theta_3) = \sin^4\theta_3\cos^2\theta_3$

$$\int_{\Omega_{\frac{\pi}{2}}} (9 + 3\sin\psi)\sigma_0^1(d\theta_1) = 0. \tag{4.226}$$

Since the integrand is positive, the measure σ_0^1 is zero on $\Omega_{\frac{\pi}{2}}$ also.

To sum up, in this case σ_0^1, is concentrated on $\Omega_{\frac{\pi}{2}}$ and its density satisfies the differential equation (4.224).

The differential equation (4.224) does not appear to have a θ_1-independent solution, and in particular $\rho(\theta_1,\theta_3) = \sin\theta_3$ is not a solution, so that there is an anomaly at $E = 1$ on the left-hand side. In the next section we will see that there is also an anomaly on the right-hand side.

4.5 The case $E \in (-3,-1) \cup (1,3)$

Suppose that $1 < E < 3$. The case $-3 < E < -1$ is similar. We can choose $\beta \in (0, \frac{\pi}{2})$ but we cannot choose α to be real number, in fact if we put $\alpha = i\gamma$, $\gamma > 0$, we get $2\cosh\gamma = E+1$ and $2\cos\beta = E-1$. Then

$$J_0 = \begin{pmatrix} R_\beta & 0 \\ 0 & \tilde{R}_\gamma \end{pmatrix}, \tag{4.227}$$

where

$$\tilde{R}_\gamma = \begin{pmatrix} \exp(-\gamma) & 0 \\ 0 & \exp(\gamma) \end{pmatrix}. \tag{4.228}$$

$$S = \begin{pmatrix} 1 & 1 & -\cos\beta & -\cos\beta \\ 0 & 0 & \sin\beta & \sin\beta \\ -\exp(-\gamma) & \exp(-\gamma) & 1 & -1 \\ \exp(\gamma) & -\exp(\gamma) & -1 & 1 \end{pmatrix} \tag{4.229}$$

and

$$S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \cot\beta & \frac{1}{2}\operatorname{cosech}\gamma & \frac{1}{2}\operatorname{cosech}\gamma \\ 1 & \cot\beta & -\frac{1}{2}\operatorname{cosech}\gamma & -\frac{1}{2}\operatorname{cosech}\gamma \\ 0 & \operatorname{cosec}\beta & \frac{1}{2}e^{\gamma}\operatorname{cosech}\gamma & \frac{1}{2}e^{-\gamma}\operatorname{cosech}\gamma \\ 0 & \operatorname{cosec}\beta & -\frac{1}{2}e^{\gamma}\operatorname{cosech}\gamma & -\frac{1}{2}e^{-\gamma}\operatorname{cosech}\gamma \end{pmatrix}. \tag{4.230}$$

We have

$$\theta_1^{(q)} = (\theta_1 - q\beta) \bmod 2\pi, \tag{4.231}$$

$$\cot\theta_2^{(q)} = \cot\theta_2 e^{2q\gamma} \tag{4.232}$$

and

$$\cot\theta_3^{(q)} = \cot\theta_3(e^{-2q\gamma}\sin^2\theta_2 + e^{2q\gamma}\cos^2\theta_2)^{\frac{1}{2}}. \tag{4.233}$$

Therefore as $q \to \infty$, $\theta_3^{(q)}$ converges to 0 or $\frac{\pi}{2}$. We have

$$\int_\Omega g(\omega)\sigma_0^E(d\omega) = \lim_{q\to\infty}\int_\Omega (\mathcal{T}_0^q g)(\omega)\sigma_0^E(d\omega). \tag{4.234}$$

By using the functions (4.33) in (4.234), we get for $n_1, n_2 \in \mathbb{Z}$, $n_3 \in \mathbb{N}$, $n_1 + n_2$ even,

$$\int_{\Omega_{(0,\frac{\pi}{2})}} e^{i(n_1\theta_1+n_2\theta_2)}\sin 2n_3\theta_3\sigma_0^E(d\theta_1 d\theta_2 d\theta_3) = 0 \tag{4.235}$$

since $\sin 2n_3\theta_3^{(q)}$ converges to 0. Thus $\sigma_0^E(\Omega_{(0,\frac{\pi}{2})}) = 0$.
If $\theta_3 = 0$ or $\frac{\pi}{2}$, then $\theta_3^{(q)} = \theta_3$. If $\theta_2 = \frac{\pi}{2}$, then $\theta_2^{(q)} = \frac{\pi}{2}$, otherwise $\theta_2^{(q)} \to 0$. So by using the functions (4.35) in (4.234), we get for $n_2 \in \mathbb{Z}$

$$\int_{\Omega_0} e^{2in_2\theta_2}\sigma_0^E(d\theta_2) = \sigma_0^E(\Omega_0 \cap \{\theta_2 \neq \frac{\pi}{2}\}) + \sigma_0^E(\Omega_0 \cap \{\theta_2 = \frac{\pi}{2}\})e^{in_2\pi}. \tag{4.236}$$

Therefore σ_0^E on is concentrated on $(\Omega_0 \cap \{\theta_2 = 0 \text{ or } \frac{\pi}{2}\}) \cup \Omega_{\frac{\pi}{2}}$.

We have

$$C_n(m) = -2\begin{pmatrix} 0 & U_n(m) \\ V_n(m) & 0 \end{pmatrix} \tag{4.237}$$

and

$$D_n(m) = 2\begin{pmatrix} \frac{1}{\sin\beta}(R_{\frac{\pi}{2}-m\beta} + R_{\frac{\pi}{2}-(2n-2-m)\beta}\sigma_z) & 0 \\ 0 & W_n(m) \end{pmatrix}. \tag{4.238}$$

We need the explicit form of $U_n(m)$, $V_n(m)$ and $W_n(m)$ only for $n = m = 1$. Note that the first entry in $D_n(m)$ is as in Section 4.3 with α replaced by β.

$$C_1(1) = 4\begin{pmatrix} 0 & 0 & \frac{1}{2\sinh\gamma} & -\frac{1}{2\sinh\gamma} \\ 0 & 0 & 0 & -\frac{1}{2\sinh\gamma} \\ -e^{-\gamma} & -e^{-\gamma}\cot\beta & 0 & 0 \\ -e^{\gamma} & -e^{\gamma}\cot\beta & 0 & 0 \end{pmatrix}, \tag{4.239}$$

$$D_1(1) = 4\begin{pmatrix} 1 & \cot\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{e^{-\gamma}}{2\sinh\gamma} & \frac{e^{-\gamma}}{2\sinh\gamma} \\ 0 & 0 & -\frac{e^{\gamma}}{2\sinh\gamma} & \frac{e^{\gamma}}{2\sinh\gamma} \end{pmatrix}. \tag{4.240}$$

From (1.24) we have

$$\lim_{\lambda\to 0}\lambda^{-2}\int_{\Omega_{\frac{\pi}{2}}} (\mathcal{T}_\lambda g - g)(\theta_1)\sigma_0^E(d\theta_1) + \lim_{\lambda\to 0}\lambda^{-2}\int_{\Omega_0\cap\{\theta_2=0\}} (\mathcal{T}_\lambda g - g)(\theta_2)\sigma_0^E(d\theta_2) = 0. \tag{4.241}$$

If we let $g(\theta_3) = \sin^2\theta_3\cos^4\theta_3$, then

$$\mathcal{T}_0 g\Big|_{(\Omega_0\cap\{\theta_2=0\})\cup\Omega_{\frac{\pi}{2}}} = 0 = g\Big|_{(\Omega_0\cap\{\theta_2=0\})\cup\Omega_{\frac{\pi}{2}}}. \tag{4.242}$$

Thus

$$\int_{\Omega_{\frac{\pi}{2}}}\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda g\right)_{\lambda=0}(\theta_1)\sigma_0^E(d\theta_1) + \int_{\Omega_0\cap\{\theta_2=0\}}\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda g\right)_{\lambda=0}(\theta_2)\sigma_0^E(d\theta_2) = 0. \tag{4.243}$$

A longish calculation using the above information and (4.65) with $n_1 = n_2 = 0$ and $m = 1$, shows that the first term is 0 and that the second term is equal to

$$16(e^{2\gamma}-1)^{-2}\sigma_0^E(\Omega_0\cap\{\theta_2=0\}) + 8(1-e^{-2\gamma})^{-2}\sigma_0^E(\Omega_0\cap\{\theta_2=\frac{\pi}{2}\}).$$

Therefore $\sigma_0^E(\Omega_0) = 0$ and σ_0^E on is concentrated on $\Omega_{\frac{\pi}{2}}$.

It is clear that $\sigma_0^E\Big|_{\Omega_{\frac{\pi}{2}}}$ is invariant under rotation by β. Thus if β/π is irrational this measure must be the Lebesgue measure. If $\beta/\pi = p/q$ where p and q are positive integers then we have

$$\int_{\Omega_{\frac{\pi}{2}}}\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda^{2q} g\right)_{\lambda=0}(\theta_1)\sigma_0^E(d\theta_1) = 0. \tag{4.244}$$

Let

$$x = \begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \\ 0 \\ 0 \end{pmatrix} \tag{4.245}$$

and let $x' = \tilde{B}(2q)x$. As in (4.65) with $n_2 = n_3 = 0$ and $m = 2q$ we have

$$\mathbb{E}\left(\exp(in_1\theta_1')\right) = \exp(in_1\theta_1)\left\{1 + \lambda^2[A_1 n_1 + A_{11}n_1^2]\right\} + O(\lambda^3). \tag{4.246}$$

Using Appendix 1 and (4.50) we get

$$\mathbb{E}(y_1^2) = \frac{1}{2}\sum_{n=1}^{2q}\left\langle D_n^T e_1, x\right\rangle^2 = \frac{2q(1+2\cos^2\theta_1)}{\sin^2\beta} \tag{4.247}$$

$$\mathbb{E}(y_1y_2) = \frac{1}{2}\sum_{n=1}^{2q}\left\langle D_n^T e_1, x\right\rangle\left\langle D_n^T e_2, x\right\rangle = -\frac{4q\sin\theta_1\cos\theta_1}{\sin^2\beta} \tag{4.248}$$

$$\mathbb{E}(y_2^2) = \frac{1}{2}\sum_{n=1}^{2q}\left\langle D_n^T e_2, x\right\rangle^2 = \frac{2q(1+2\sin^2\theta_1)}{\sin^2\beta}. \tag{4.249}$$

One can then check that when $\theta_3 = \frac{\pi}{2}$,

$$A_1 = 0, \qquad \text{and} \qquad A_{11} = -\frac{3q}{\sin^2\beta}. \tag{4.250}$$

Therefore from 4.244 for $n_1 \neq 0$

$$\int_{\Omega_{\frac{\pi}{2}}} e^{2in_1\theta_1}\sigma_0^E(d\theta_1) = 0. \tag{4.251}$$

Thus $\sigma_0^E\big|_{\Omega_{\frac{\pi}{2}}}$ is Lebesgue measure.

Summing up, we have that σ_0^E is concentrated on $\Omega_{\frac{\pi}{2}}$ and on that it is Lebesgue measure.

4.6 The cases $E = \pm 3$

Finally we come to the case $E = \pm 3$. It is sufficient to study the case $E = 3$. Here $\cos\alpha = 2$ and $\beta = 0$.

$$J_0 = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}, \tag{4.252}$$

where

$$\mathcal{J}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.253}$$

and

$$\mathcal{J}_2 = \begin{pmatrix} 2-\sqrt{3} & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix}. \tag{4.254}$$

$$S_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ -(2-\sqrt{3}) & 2-\sqrt{3} & 1 & -1 \\ 2+\sqrt{3} & -(2+\sqrt{3}) & -1 & 1 \end{pmatrix}. \tag{4.255}$$

We have

$$\cot\theta_1^{(q)} = \begin{cases} \frac{1}{q} & \text{if } \theta_1 = 0, \\ \frac{\cot\theta_1}{1+q\cot\theta_1}, & \text{if } \theta_1 \neq 0, \end{cases} \tag{4.256}$$

$$\cot\theta_2^{(q)} = \cot\theta_2(2+\sqrt{3})^{2q} \tag{4.257}$$

and

$$\cot\theta_3^{(q)} = \cot\theta_3\left(\frac{(2-\sqrt{3})^{2q}\sin^2\theta_2 + (2+\sqrt{3})^{2q}\cos^2\theta_2}{1+q^2\cos^2\theta_1 + 2q\sin\theta_1\cos\theta_1}\right)^{\frac{1}{2}}. \tag{4.258}$$

Therefore as $q \to \infty$, $\theta_3^{(q)} \to 0$ or $\frac{\pi}{2}$. We have

$$\int_\Omega g(\omega)\sigma_0^3(d\omega) = \lim_{q\to\infty}\int_\Omega (\mathcal{T}_0^q g)(\omega)\sigma_0^3(d\omega). \tag{4.259}$$

By using the functions (4.33) in (4.259), we get for $n_1, n_2 \in \mathbb{Z}$, $n_3 \in \mathbb{N}$, $n_1 + n_2$ even,

$$\int_{\Omega_{(0,\frac{\pi}{2})}} e^{i(n_1\theta_1+n_2\theta_2)} \sin 2n_3\theta_3\sigma_0^3(d\theta_1 d\theta_2 d\theta_3) = 0. \tag{4.260}$$

Thus $\sigma_0^3(\Omega_{(0,\frac{\pi}{2})}) = 0$.

Now $\theta_3^{(q)} = \theta_3$ if $\theta_3 = 0$ or $\frac{\pi}{2}$. $\theta_2^{(q)} = \theta_2$ if $\theta_2 = 0$ or $\frac{\pi}{2}$, otherwise $\theta_2^{(q)} \to 0$ as $q \to \infty$. Therefore by the same argument as in Section 4.4, σ_0^3 on is concentrated on $(\Omega_0 \cap \{\theta_2 = 0 \text{ or } \theta_2 = \frac{\pi}{2}\}) \cup \Omega_{\frac{\pi}{2}}$.

Similarly, since $\theta_1^{(q)} \to \frac{\pi}{2}$ as $q \to \infty$, we can argue that σ_0^3 on is concentrated on $(\Omega_0 \cap \{\theta_2 = 0 \text{ or } \theta_2 = \frac{\pi}{2}\}) \cup (\Omega_{\frac{\pi}{2}} \cap \{\theta_1 = \frac{\pi}{2}\})$.

Using the notation of Section 4.2 we have

$$C_1(1) = \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -(2-\sqrt{3}) & 0 & 0 & 0 \\ 2+\sqrt{3} & 0 & 0 & 0 \end{pmatrix}, \tag{4.261}$$

$$D_1(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2-\sqrt{3}}{2\sqrt{3}} & -\frac{2-\sqrt{3}}{2\sqrt{3}} \\ 0 & 0 & \frac{2+\sqrt{3}}{2\sqrt{3}} & \frac{2+\sqrt{3}}{2\sqrt{3}} \end{pmatrix}. \tag{4.262}$$

From (1.24) we have

$$\lim_{\lambda\to 0} \lambda^{-2} \int_{\Omega_{\frac{\pi}{2}} \cap \{\theta_1 = \frac{\pi}{2}\}} (\mathcal{T}_\lambda g - g)(\theta_1)\sigma_0^E(d\theta_1) + \lim_{\lambda\to 0} \lambda^{-2} \int_{\Omega_0 \cap \{\theta_2 = 0 \text{ or } \frac{\pi}{2}\}} (\mathcal{T}_\lambda g - g)(\theta_2)\sigma_0^E(d\theta_2) = 0. \tag{4.263}$$

If we let $g(\theta_3) = \sin^2\theta_3 \cos^4\theta_3$, then

$$\mathcal{T}_0 g\Big|_{\Omega_0 \cup \Omega_{\frac{\pi}{2}}} = 0 = g\Big|_{\Omega_0 \cup \Omega_{\frac{\pi}{2}}}. \tag{4.264}$$

Thus

$$\int_{\Omega_{\frac{\pi}{2}} \cap \{\theta_1 = \frac{\pi}{2}\}} \left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda g\right)_{\lambda=0}(\theta_1)\sigma_0^E(d\theta_1) + \int_{\Omega_0 \cap \{\theta_2 = 0 \text{ or } \frac{\pi}{2}\}} \left(\frac{\partial^2}{\partial\lambda^2}\mathcal{T}_\lambda g\right)_{\lambda=0}(\theta_2)\sigma_0^E(d\theta_2) = 0. \tag{4.265}$$

From the above information and (4.65) with $n_1 = n_2 = 0$ and $m = 1$, we can check that the first term is 0 and that the second term is equal to $\sigma_0^E(\Omega_0 \cap \{\theta_2 = 0 \text{ or } \frac{\pi}{2}\})/12$. Therefore $\sigma_0^E(\Omega_0 \cap \{\theta_2 = 0 \text{ or } \frac{\pi}{2}\}) = 0$ and σ_0^E is concentrated on $\Omega_{\frac{\pi}{2}} \cap \{\theta_1 = \frac{\pi}{2}\}$.

Thus σ_0^E is concentrated on $\Omega_{\frac{\pi}{2}}$ and on that it is the atomic measure at $\theta_1 = \frac{\pi}{2}$.

The limiting measure at $E = 1$ has to be transformed via the matrix

$$S_3\, S^{-1} = \begin{pmatrix} 1 & \cot\beta & 0 & 0 \\ 1 & \frac{\cos\beta - 1}{\sin\beta} & 0 & 0 \\ 0 & 0 & \frac{-2+\sqrt{3}+e^{\gamma}}{2\sinh\gamma} & \frac{-2+\sqrt{3}+e^{-\gamma}}{2\sinh\gamma} \\ 0 & 0 & \frac{2+\sqrt{3}-e^{\gamma}}{2\sinh\gamma} & \frac{2+\sqrt{3}-e^{-\gamma}}{2\sinh\gamma} \end{pmatrix}. \tag{4.266}$$

This matrix transforms the point $(1,0,0,0)$ to $(1,1,0,0)$, so that the transformed measure is concentrated on $\Omega_{\pi/2} \cap \{\theta_1' = \pi/4\}$.

5 Appendix 1 The expectation of the terms $y_i y_j$

Case $i = 1, j = 1$.

$$\begin{aligned}\sum_{n=1}^{m}\left\langle D_n^T e_1, x\right\rangle^2 &= 2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2\Big\{m\cos(2m\alpha-2\theta_1)+2m\\ &\quad+\cos(2\alpha-2\theta_1)\frac{\sin 2m\alpha}{\sin 2\alpha}\\ &\quad+4\cos(m\alpha-\theta_1)\cos(\alpha-\theta_1)\frac{\sin m\alpha}{\sin\alpha}\Big\}\end{aligned} \tag{5.1}$$

$$\begin{aligned}\sum_{n=1}^{m}\left\langle C_n^T e_1, x\right\rangle^2 &= 2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2\Big\{\cos[(m-1)(\alpha+\beta)-2\theta_2]\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)}\\ &\quad+\cos[(m-1)(\alpha-\beta)+2\theta_2]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)}\\ &\quad+2\cos((m-1)\alpha)\frac{\sin m\alpha}{\sin\alpha}\\ &\quad+2\cos[(m-1)\beta-2\theta_2]\frac{\sin m\beta}{\sin\beta}+2m\Big\}\end{aligned} \tag{5.2}$$

Case $i = 1, j = 2$.

$$\begin{aligned}\sum_{n=1}^{m}\left\langle D_n^T e_1, x\right\rangle\left\langle D_n^T e_2, x\right\rangle &= 2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2\Big\{m\sin(2m\alpha-2\theta_1)\\ &\quad-\sin(2\alpha-2\theta_1)\frac{\sin 2m\alpha}{\sin 2\alpha}\\ &\quad+2\sin((m-1)\alpha)\frac{\sin m\alpha}{\sin\alpha}\Big\}\end{aligned} \tag{5.3}$$

$$\begin{aligned}\sum_{n=1}^{m}\left\langle C_n^T e_1, x\right\rangle\left\langle C_n^T e_2, x\right\rangle &= 2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2\Big\{\sin[(m-1)(\alpha+\beta)-2\theta_2]\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)}\\ &\quad+\sin[(m-1)(\alpha-\beta)+2\theta_2]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)}\\ &\quad+2\sin((m-1)\alpha)\frac{\sin m\alpha}{\sin\alpha}\Big\}\end{aligned} \tag{5.4}$$

Case $i = 1, j = 3$.

$$\begin{aligned}
\sum_{n=1}^{m}\left\langle D_n^T e_1, x\right\rangle\left\langle D_n^T e_3, x\right\rangle &= \sum_{n=1}^{m}\left\langle C_n^T e_1, x\right\rangle\left\langle C_n^T e_3, x\right\rangle \\
&= 2\frac{\sin\theta_3\ \cos\theta_3}{\sin\alpha\ \sin\beta}\Big\{ m\cos[m(\alpha+\beta)-\theta_1-\theta_2] \\
&\quad +m\cos[m(\alpha-\beta)-\theta_1+\theta_2] \\
&\quad +\cos(\alpha-\beta-\theta_1-\theta_2)\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\
&\quad +\cos(\alpha+\beta-\theta_1+\theta_2)\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\
&\quad +2\cos(m\beta-\theta_2)\cos(\alpha-\theta_1)\frac{\sin m\alpha}{\sin\alpha} \\
&\quad +2\cos(m\alpha-\theta_1)\cos(\beta+\theta_2)\frac{\sin m\beta}{\sin\beta}\Big\} \qquad (5.5)
\end{aligned}$$

Case $i=1, j=4$.

$$\begin{aligned}
\sum_{n=1}^{m}\left\langle D_n^T e_1, x\right\rangle\left\langle D_n^T e_4, x\right\rangle &= \sum_{n=1}^{m}\left\langle C_n^T e_1, x\right\rangle\left\langle C_n^T e_4, x\right\rangle \\
&= 2\frac{\sin\theta_3\ \cos\theta_3}{\sin\alpha\ \sin\beta}\Big\{ m\sin[m(\alpha+\beta)-\theta_1-\theta_2] \\
&\quad -m\sin(m(\alpha-\beta)-\theta_1+\theta_2) \\
&\quad -\sin(\alpha-\beta-\theta_1-\theta_2)\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\
&\quad +\sin(\alpha+\beta-\theta_1+\theta_2)\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\
&\quad +2\cos(\alpha-\theta_1)\sin(m\beta-\theta_2)\frac{\sin m\alpha}{\sin\alpha} \\
&\quad +2\cos(m\alpha-\theta_1)\sin(\beta+\theta_2)\frac{\sin m\beta}{\sin\beta}\Big\} \qquad (5.6)
\end{aligned}$$

Case $i=2, j=2$.

$$\begin{aligned}
\sum_{n=1}^{m}\left\langle D_n^T e_2, x\right\rangle^2 &= -2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2\Big\{ m\cos(2m\alpha-2\theta_1)-2m \\
&\quad +\cos(2\alpha-2\theta_1)\frac{\sin 2m\alpha}{\sin 2\alpha} \\
&\quad +4\sin(m\alpha-\theta_1)\sin(\alpha-\theta_1)\frac{\sin m\alpha}{\sin\alpha}\Big\} \qquad (5.7)
\end{aligned}$$

$$\sum_{n=1}^{m}\left\langle C_n^T e_2, x\right\rangle^2 = -2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2\Big\{ \cos((m-1)(\alpha+\beta)-2\theta_2)\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)}$$

$$
\begin{aligned}
&+\cos[(m-1)(\alpha-\beta)+2\theta_2]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\
&+2\cos((m-1)\alpha)\frac{\sin m\alpha}{\sin\alpha} \\
&-2\cos[(m-1)\beta-2\theta_2]\frac{\sin m\beta}{\sin\beta}-2m\Bigg\}
\end{aligned}
\tag{5.8}
$$

Case $i=2, j=3$.

$$
\begin{aligned}
\sum_{n=1}^{m}\left\langle D_n^T e_2, x\right\rangle\left\langle D_n^T e_3, x\right\rangle &= \sum_{n=1}^{m}\left\langle C_n^T e_2, x\right\rangle\left\langle C_n^T e_3, x\right\rangle \\
&= 2\frac{\sin\theta_3\,\cos\theta_3}{\sin\alpha\,\sin\beta}\Bigg\{m\sin[m(\alpha+\beta)-\theta_1-\theta_2] \\
&\quad +m\sin[m(\alpha-\beta)-\theta_1+\theta_2] \\
&\quad -\sin(\alpha-\beta-\theta_1-\theta_2)\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\
&\quad -\sin(\alpha+\beta-\theta_1+\theta_2)\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\
&\quad -2\sin(\alpha-\theta_1)\cos(m\beta-\theta_2)\frac{\sin m\alpha}{\sin\alpha} \\
&\quad +2\sin(m\alpha-\theta_1)\cos(\beta+\theta_2)\frac{\sin m\beta}{\sin\beta}\Bigg\}
\end{aligned}
\tag{5.9}
$$

Case $i=2, j=4$.

$$
\begin{aligned}
\sum_{n=1}^{m}\left\langle D_n^T e_2, x\right\rangle\left\langle D_n^T e_4, x\right\rangle &= \sum_{n=1}^{m}\left\langle C_n^T e_2, x\right\rangle\left\langle C_n^T e_4, x\right\rangle \\
&= -2\frac{\sin\theta_3\,\cos\theta_3}{\sin\alpha\,\sin\beta}\Bigg\{m\cos[m(\alpha+\beta)-\theta_1-\theta_2] \\
&\quad -m\cos[m(\alpha-\beta)-\theta_1+\theta_2] \\
&\quad +\cos(\alpha-\beta-\theta_1-\theta_2)\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\
&\quad -\cos(\alpha+\beta-\theta_1+\theta_2)\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\
&\quad +2\sin(\alpha-\theta_1)\sin(m\beta-\theta_2)\frac{\sin m\alpha}{\sin\alpha} \\
&\quad -2\sin(m\alpha-\theta_1)\sin(\beta+\theta_2)\frac{\sin m\beta}{\sin\beta}\Bigg\}
\end{aligned}
\tag{5.10}
$$

Case $i=3, j=3$.

$$\begin{aligned}\sum_{n=1}^{m}\left\langle D_n^T e_3, x\right\rangle^2 &= 2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2\Big\{m\cos(2m\beta-2\theta_2)+2m\\&+\cos(2\beta+2\theta_2)\frac{\sin 2m\beta}{\sin 2\beta}\\&+4\cos(m\beta-\theta_2)\cos(\beta+\theta_2)\frac{\sin m\beta}{\sin\beta}\Big\}\end{aligned}\tag{5.11}$$

$$\begin{aligned}\sum_{n=1}^{m}\left\langle C_n^T e_3, x\right\rangle^2 &= 2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2\Big\{\cos[(m+1)(\alpha+\beta)-2\theta_1]\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)}\\&+\cos[(m+1)(\alpha-\beta)-2\theta_1]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)}\\&+2\cos[(m+1)\alpha-2\theta_1]\frac{\sin m\alpha}{\sin\alpha}\\&+2\cos[(m+1)\beta]\frac{\sin m\beta}{\sin\beta}+2m\Big\}\end{aligned}\tag{5.12}$$

Case $i = 3, j = 4$.

$$\begin{aligned}\sum_{n=1}^{m} \left\langle D_n^T e_3, x\right\rangle \left\langle D_n^T e_4, x\right\rangle &= 2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2 \Big\{ m\sin(2m\beta - 2\theta_2) \\ &+ \sin(2\beta + 2\theta_2)\frac{\sin 2m\beta}{\sin 2\beta} \\ &+ 2\sin((m+1)\beta)\frac{\sin m\beta}{\sin\beta}\Big\} \end{aligned} \tag{5.13}$$

$$\begin{aligned}\sum_{n=1}^{m} \left\langle C_n^T e_3, x\right\rangle \left\langle C_n^T e_4, x\right\rangle &= 2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2 \Big\{ \sin[(m+1)(\alpha+\beta) - 2\theta_1]\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\ &- \sin[(m+1)(\alpha-\beta) - 2\theta_1]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\ &+ 2\sin((m+1)\beta)\frac{\sin m\beta}{\sin\beta}\Big\} \end{aligned} \tag{5.14}$$

Case $i = 4, j = 4$.

$$\begin{aligned}\sum_{n=1}^{m} \left\langle D_n^T e_4, x\right\rangle^2 &= -2\left(\frac{\cos\theta_3}{\sin\beta}\right)^2 \Big\{ m\cos(2m\beta - 2\theta_2) - 2m \\ &+ \cos(2\beta + 2\theta_2)\frac{\sin 2m\beta}{\sin 2\beta} \\ &- 4\sin(m\beta - \theta_2)\sin(\beta + \theta_2)\frac{\sin m\beta}{\sin\beta}\Big\} \end{aligned} \tag{5.15}$$

$$\begin{aligned}\sum_{n=1}^{m} \left\langle C_n^T e_4, x\right\rangle^2 &= -2\left(\frac{\sin\theta_3}{\sin\alpha}\right)^2 \Big\{ \cos[(m+1)(\alpha+\beta) - 2\theta_1]\frac{\sin m(\alpha-\beta)}{\sin(\alpha-\beta)} \\ &+ \cos[(m+1)(\alpha-\beta) - 2\theta_1]\frac{\sin m(\alpha+\beta)}{\sin(\alpha+\beta)} \\ &- 2\cos[(m+1)\alpha - 2\theta_1]\frac{\sin m\alpha}{\sin\alpha} \\ &+ 2\cos[(m+1)\beta]\frac{\sin m\beta}{\sin\beta} - 2m\Big\} \end{aligned} \tag{5.16}$$

6 Appendix 2 Continuity etc

If M is an $n \times n$ matrix with $\det M = \pm 1$ then

$$\|Mx\| \geq \frac{\|x\|}{n!\|M\|^{(n-1)}}. \tag{6.1}$$

This follows from the inequality

$$\|M^{-1}\| \le n!\frac{\|M\|^{(n-1)}}{|\det M|} = n!\|M\|^{(n-1)}, \tag{6.2}$$

which gives

$$\|x\| = \|M^{-1}Mx\| \le \|M^{-1}\|\|Mx\| \le n!\|M\|^{(n-1)}\|Mx\|. \tag{6.3}$$

Let $M(\lambda)$ be a 2×2 matrix with $\det M(\lambda) = \pm 1$. Let

$$f_\lambda(x) = \tan^{-1}\frac{x_1'}{x_2'}, \tag{6.4}$$

where $x' = M(\lambda)x$ and let $M^{(r)} \equiv \dfrac{\partial^r M(\lambda)}{\partial\lambda^r}$. Then

$$\frac{\partial}{\partial\lambda}f_\lambda(x) = \frac{M^{(1)}(\lambda)x \wedge M(\lambda)x}{\|M(\lambda)x\|^2} \tag{6.5}$$

and so

$$\left|\frac{\partial}{\partial\lambda}f_\lambda(x)\right| \le \frac{\|M^{(1)}(\lambda)x\|}{\|M(\lambda)x\|}. \tag{6.6}$$

Similarly

$$\left|\frac{\partial^2}{\partial\lambda^2}f_\lambda(x)\right| \le \frac{\|M^{(2)}(\lambda)x\|}{\|M(\lambda)x\|} + 2\frac{\|M^{(1)}(\lambda)x\|^2}{\|M(\lambda)x\|^2} \tag{6.7}$$

and

$$\left|\frac{\partial^3}{\partial\lambda^3}f_\lambda(x)\right| \le \frac{\|M^{(3)}(\lambda)x\|}{\|M(\lambda)x\|} + 7\frac{\|M^{(1)}(\lambda)x\|\|M^{(2)}(\lambda)x\|}{\|M(\lambda)x\|^2} + 10\frac{\|M^{(1)}(\lambda)x\|^3}{\|M(\lambda)x\|^3}. \tag{6.8}$$

In general

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \le \sum_{\substack{r_1+r_2+\ldots+r_n=k\\ 1\le r_i\le k}} C_{r_1,\ldots,r_n}\frac{\|M^{(r_1)}(\lambda)x\|\ldots\|M^{(r_n)}(\lambda)x\|}{\|M(\lambda)x\|^n} \tag{6.9}$$

and therefore

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \le \sum_{\substack{r_1+r_2+\ldots+r_n=k\\ 1\le r_i\le k}} C_{r_1,\ldots,r_n}2^n\|M^{(r_1)}(\lambda)\|\ldots\|M^{(r_n)}(\lambda)\|\|M(\lambda)\|^n. \tag{6.10}$$

Now we take $M(\lambda) = \prod_{n=1}^q D_\lambda^{(n)}$ where $D_\lambda^{(n)} = SA_\lambda^{(n)}S^{-1}$. Note that $\dfrac{\partial^2 D_\lambda^{(n)}}{\partial\lambda^2} = 0$ and if the random variables X_n's are bounded then there exists a constant C such that both $\|D_\lambda^{(n)}\|$ and $\left\|\dfrac{\partial D_\lambda^{(n)}}{\partial\lambda}\right\|$ are bounded by C for all n and all $\lambda\in[-1,1]$. Therefore (6.10) gives for any $k\in\mathbb{N}$,

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \le C_k. \tag{6.11}$$

If $h_\lambda = g \circ f_\lambda$, where the first k derivatives of g are bounded, then we also have

$$\left|\frac{\partial^k}{\partial\lambda^k} h_\lambda(x)\right| \leq K_k. \tag{6.12}$$

Since $\mathcal{T}_\lambda^q g = \mathbb{E}(h_\lambda \circ t^{-1})$, this gives

$$\left\|\left(\frac{\partial^k}{\partial\lambda^k}\mathcal{T}_\lambda^q g\right)\right\| \leq K_k. \tag{6.13}$$

By using the Mean-Value Theorem we then see that if the first $r+1$ derivatives of g are bounded, then we also have

$$\lim_{\lambda\to 0}\lambda^{-r}\left\|\mathcal{T}_\lambda^q g - \sum_{k=0}^{r}\frac{\lambda^k}{k!}\left(\frac{\partial^k}{\partial\lambda^k}\mathcal{T}_\lambda^q g\right)_{\lambda=0}\right\| = 0 \tag{6.14}$$

Now let $M(\lambda)$ be a 4×4 matrix with $\det M(\lambda) = \pm 1$ and let

$$f_\lambda(x) = \frac{{x'_1}^2 + {x'_2}^2}{{x'_1}^2 + {x'_2}^2 + {x'_1}^3 + {x'_4}^2}, \tag{6.15}$$

where $x' = M(\lambda)x$, that is

$$f_\lambda(x) = \frac{\|PM(\lambda)x\|^2}{\|M(\lambda)x\|^2}, \tag{6.16}$$

where $Px = (x_1, x_2, 0, 0)$ or in the notation of Section 4.2, $f_\lambda(x) = \sin^2(\theta'_3)$. Then

$$\frac{\partial}{\partial\lambda}f_\lambda(x) = \frac{(PM^{(1)}(\lambda)x\cdot PM(\lambda)x)\|M(\lambda)x\|^2 - (M^{(1)}(\lambda)x\cdot M(\lambda)x)\|PM(\lambda)x\|^2}{\|M(\lambda)x\|^4} \tag{6.17}$$

and so

$$\left|\frac{\partial}{\partial\lambda}f_\lambda(x)\right| \leq 2\frac{\|M(\lambda)x\|\|M^{(1)}(\lambda)x\|}{\|M(\lambda)x\|^2}. \tag{6.18}$$

In general

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \leq \sum_{\substack{r_1+r_2+\ldots+r_n=k\\ 1\leq r_i\leq k}} C_{r_1,\ldots,r_n}\frac{\|M^{(r_1)}(\lambda)x\|\ldots\|M^{(r_n)}(\lambda)x\|}{\|M(\lambda)x\|^n} \tag{6.19}$$

and therefore

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \leq \sum_{\substack{r_1+r_2+\ldots+r_n=k\\ 1\leq r_i\leq k}} C_{r_1,\ldots,r_n}(4!)^n\|M^{(r_1)}(\lambda)\|\ldots\|M^{(r_n)}(\lambda)\|\|M(\lambda)\|^{3n}. \tag{6.20}$$

Again if we take $M(\lambda) = \prod_{n=1}^q D_\lambda^{(n)}$ we get for any $k\in\mathbb{N}$,

$$\left|\frac{\partial^k}{\partial\lambda^k}f_\lambda(x)\right| \leq C_k. \tag{6.21}$$

If $h_\lambda = l \circ f_\lambda$, where the first k derivatives of l are bounded, then we also have

$$\left|\frac{\partial^k}{\partial\lambda^k}h_\lambda(x)\right| \leq K_k. \tag{6.22}$$

If $g(\theta_3) = l(\sin^2\theta_3)$ then $\mathcal{T}_\lambda^q g = \mathbb{E}(h_\lambda \circ t^{-1})$ and we get

$$\left\| \left(\frac{\partial^k}{\partial \lambda^k} \mathcal{T}_\lambda^q g \right) \right\| \leq K_k. \tag{6.23}$$

By using the Mean-Value Theorem we then see that if the first $r+1$ derivatives of l are bounded, then we also have

$$\lim_{\lambda \to 0} \lambda^{-r} \left\| \mathcal{T}_\lambda^q g - \sum_{k=0}^{r} \frac{\lambda^k}{k!} \left(\frac{\partial^k}{\partial \lambda^k} \mathcal{T}_\lambda^q g \right)_{\lambda=0} \right\| = 0 \tag{6.24}$$

Now we want to consider functions of the form $e^{iN\theta_1'} \sin^{2s}\theta_3'$. First let

$$t_\lambda(x) = \tan^{-1}\left(\frac{x_1}{x_2}\right), \tag{6.25}$$

that is $t_\lambda(x) = \theta_1'$. Then as in (6.9)

$$\left| \frac{\partial^k}{\partial \lambda^k} t_\lambda(x) \right| \leq \sum_{\substack{r_1+r_2+\ldots+r_n=k \\ 1 \leq r_i \leq k}} C_{r_1,\ldots,r_n} \frac{\|PM^{(r_1)}(\lambda)x\| \ldots \|PM^{(r_n)}(\lambda)x\|}{\|PM(\lambda)x\|^n} \tag{6.26}$$

Let $S_\lambda(x) = \exp(iN\, t_\lambda(x))(f_\lambda(x))^s$ where f_λ is as in (6.16), that is, $S_\lambda(x) = e^{iN\theta_1'} \sin^{2s}\theta_3'$. $\frac{\partial^k}{\partial \lambda^k} S_\lambda(x)$ consists of a finite linear combination of terms with $l = 0, \ldots, k$, of the form

$$\exp(iN\, t_\lambda(x))(f_\lambda(x))^{(s-n)} \left(f_\lambda^{(p_1)}(x) \ldots f_\lambda^{(p_n)}(x) \right) \left(t_\lambda^{(q_1)}(x) \ldots t_\lambda^{(q_m)}(x) \right) \tag{6.27}$$

with $p_1 + \ldots + p_n = l$, $n \leq l$ and $q_1 + \ldots + q_m = k - l$, $m \leq k - l$. If we use (6.26) to get an upper bound for $|t_\lambda^{(q_1)}(x) \ldots t_\lambda^{(q_m)}(x)|$ we see that the highest power of $\|PM(\lambda)x\|$ in the denominator of the upper bound is $k - l$. From (6.16) we see that the term in 6.27 is bounded if $s - n \geq (k-l)/2$ and therefore if $s \geq (k+l)/2$ it is bounded for all m. Thus if $s \geq k$, $\frac{\partial^k}{\partial \lambda^k} S_\lambda(x)$ is bounded.

Clearly the same argument works for $e^{iM\theta_2'} \cos^{2s}\theta_3'$ and for $e^{i(N\theta_1' + M\theta_2')} \sin^{2s_1}\theta_3' \cos^{2s_2}\theta_3'$ if $s_1 \geq k$ and $s_2 \geq k$.

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